New Large Sets of $t-$Designs

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Abstract

We introduce generalizations of earlier direct methods for constructing large sets of $t-$designs. These are based on assembling systematically orbits of $t-$homogeneous permutation groups in their induced actions on $k-$subsets. By means of these techniques and the known recursive methods we construct an extensive number of new large sets, including new infinite families. In particular, a new series of $LS[3](2(2+m), 6\cdot 3^m - 2, 16\cdot 3^m - 2)$ is obtained. This also provides the smallest known v for a $t-(v,k,\lambda)$ design when $t \geq 16$. We present our results compactly for $v \leq 61$, in tables derived from Pascal's triangle modulo appropriate primes.

1 Introduction

Large sets of t-designs have been used in recursive constructions for over a decade. Most celebrated is the pioneering work of Teirlinck who showed in [26] that simple t-designs exist for all t. Unfortunately, for a given t, Teirlinck's constructions result in t-designs with extremely large values for the parameters v and λ. Subsequently, other researchers, particularly Khosrovshahi and Ajoodani-Namini, greatly contributed to the repertory of recursive methods. By means of these techniques the value of v for which t-designs are now constructible can be made considerably smaller.

Recursive methods require a basis of large sets from which to start. Then infinite series of parameter sets are settled by recursion. In this article, we present several new large sets which, in combination with already known cases and the known recursive methods, handle many admissible parameter sets.

By an admissible parameter set for a putative $LS[N](t,k,v)$ we mean parameters (N,t,k,v) which satisfy certain obvious divisibility conditions as discussed in the next section. In the case of halvings, i. e. $LS[2](t,k,v)$, Ajoodani-Namini [1] has shown that for $t=2$ all admissible parameter sets are realizable. For larger t there are partial results for k up to 16 [21]. We review what is presently known by means of the following theorem.

Theorem 1 *Let p be an odd prime, and $0 < t < k < v$ be integers. Then, large sets $LS[p](t,k,v)$ are realizable as follows:*

i) For all $k \leq 11$ all admissible $LS[3](2,k,v)$ are realizable.

ii) For all $k \leq 8$ all admissible $LS[3](3,k,v)$ are realizable.

iii) For all $k \leq 8$ all admissible $LS[3](4,k,v)$ are realizable.

iv) For all $k \leq 5$ all admissible $LS[5](2,k,v)$ are realizable, with the well known exceptions of an $LS[5](2,3,7)$ and an $LS[5](2,4,7)$.

v) For all $k \leq 5$ all admissible $LS[5](3,k,v)$ are realizable, with the well known exception of an $LS[5](3,4,8)$.

vi) For all $k \leq 6$ all admissible $LS[7](2,k,v)$ are realizable.

vii) For all $k \leq 10$ all admissible $LS[11](2,k,v)$ are realizable.

viii) For all $k \leq 5$ all admissible $LS[29](2,k,v)$ are realizable.

There exist $LS[3](2(2+m), 6\cdot 3^m - 2, 16\cdot 3^m - 2)$ for all natural numbers m. This provides the smallest known v when $t \geq 16$.

Our results are presented for $v \leq 61$ in tables that are deduced from Pascal's triangle, making it easy to depict unsettled cases. Disposing of the unsettled cases will either require new recursive techniques or direct construction methods different from the ones employed in this paper.

The large sets in this paper are obtained by appropriately assembling orbits of t-homogeneous groups. This approach had been employed earlier, see [12], with the additional restriction that all orbits of k-sets were of length equal to the group order. In this paper we generalize the approach by also considering situations where orbits of k-sets of various lengths can occur. In some cases, a random search for disjoint t-designs contributes large sets from non t-homogeneous group actions. The computations here were made using DISCRETA, a software package developed at Bayreuth University, as well as a special purpose computer program written for this article.

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2 Preliminaries

In this paper, V denotes a finite point set with $|V| = v$, t and k are positive integers such that $0 < t < k \leq v$, and the collection of all k-subsets of V is denoted by $\binom{V}{k}$.

A simple $t-(v,k,\lambda)$ design, $(V, \mathcal{B})$, is a v-element set V of *points* and a collection $\mathcal{B}$ of k-element subsets of V called *blocks*, such that every t-element subset of V is contained in precisely λ blocks. All $t-(v,k,\lambda)$ designs discussed in this paper are simple.

If $(V, \mathcal{B})$ is a $t-(v,k,\lambda)$ design, and $x \in V$, the *derived* design with respect to x is $(V \setminus \{x\}, \mathcal{D})$, where $D \in \mathcal{D}$ if and only if $D = B \setminus \{x\}$, for $x \in B \in \mathcal{B}$. A derived design is a $(t-1)-(v-1,k-1,\lambda)$ design.

If $(V, \mathcal{B})$ is a $t-(v,k,\lambda)$ design, and $x \in V$, the *residual* design with respect to x is the design $(V \setminus \{x\}, \mathcal{R})$, where $K \in \mathcal{R}$ if and only if $x \notin K \in \mathcal{B}$. A residual design is a $(t-1)-(v-1,k,\lambda')$ design.

It is well known that for each s, $0 \leq s \leq t$, every $t-(v,k,\lambda)$ design is also an $s-(v,k,\lambda_s)$ design, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Thus, a set of necessary *divisibility conditions* for the existence of a $t-(v,k,\lambda)$ design is that $\lambda\binom{v-s}{t-s} \equiv 0 \bmod \binom{k-s}{t-s}$, for $0 \leq s < t$.

By a *large set* $\mathrm{LS}[N](t,k,v)$ we mean a collection $\mathcal{L} = \{(V, \mathcal{B}_i)\}_{i=1}^{N}$ of $t-(v,k,\lambda)$ designs where $\{\mathcal{B}_i\}_{i=1}^{N}$ is a partition of $\binom{V}{k}$.

The number of blocks in a $t-(v,k,\lambda)$ design is $b = \lambda_0 = \lambda\binom{v}{t}/\binom{k}{t}$. Thus, a necessary condition for a large set $\mathrm{LS}[N](t,k,v)$ to exist is that $Nb = \binom{v}{k}$. This is equivalent to $\lambda N = \binom{v-t}{k-t}$. Thus, N must divide $\binom{v-t}{k-t}$.

A group action $G|V$ is called *transitive* if V consists of a single G-orbit. The group action $G|V$ is said to be t-*homogeneous* if the induced action of G on $\binom{V}{t}$ is transitive. For brevity, by a $k-$orbit we mean an orbit of G in its induced action on $\binom{V}{k}$.

Let $\mathbb{B} = \{\mathcal{B}_i\}_{i=1}^{N}$ be the collection of designs in a large set $\mathcal{L}$. A group G is said to be an automorphism group of $\mathcal{L}$ if $\mathbb{B}^g$=$\mathbb{B}$ for all $g \in G$, that is, if $\mathcal{B}_i^g \in \mathbb{B}$ for all $\mathcal{B}_i \in \mathbb{B}$ and $g \in G$. Equivalently, we say that a large set with this property is G-invariant. If the stronger condition that $\mathcal{B}_i^g$=$\mathcal{B}_i$ for all $\mathcal{B}_i \in \mathbb{B}$ and $g \in G$ holds, we say that a large set is $[G]$-invariant.

In 1976, Kramer and Mesner [24] described certain matrix invariants $A_{t,k}$ associated with a given group action $G|V$. Roughly speaking $A_{t,k}$ is the result of fusing under G the incidence matrix between $\binom{V}{t}$ and $\binom{V}{k}$, where incidence is set inclusion. More precisely, for a given group action $G|V$, let $\Delta = \{\Delta_i\}_{i=1}^{r}$ be the collection of G-orbits on $\binom{V}{t}$, and $\Gamma = \{\Gamma_j\}_{j=1}^{s}$ be the collection of G-orbits on $\binom{X}{k}$. For a fixed member T of Δ_i, the number $a_{ij}(T)$ of members $K \in \Gamma_j$ such that $T \subset K$ is independent of the choice of $T \in \Delta_i$, hence we may write $a_{ij} = a_{ij}(T)$. We define the $r \times s$ matrix $A_{t,k} = A_{t,k}(G|X)$ by $A_{t,k} = (a_{ij})$.

In [24] Kramer and Mesner state a theorem which provides necessary and sufficient conditions for the existence of a $G-$invariant $t-(v,k,\lambda)$ design in terms of the matrix $A_{t,k}$ above. Beginning with a given group action $G|V$, the theorem allows for the construction of all such $G-$invariant $t-$designs. In [12] the authors describe a slight generalization of the theorem which provides means for constructing $[G]$-invariant *large sets* of $t-(v,k,\lambda)$ designs. In particular, the authors of [12] turn their attention to $t-$homogeneous, G-semiregular large sets of $t-$designs.

3 Direct Constructions

The methods in this section are based on the concept of assembling orbits of a permutation group into t-designs so that these designs form a large set.

We use the Kramer-Mesner method to find t-designs from the orbits of a permutation group $G|V$ in its induced action on k-subsets. Thus, we may first construct disjoint designs with various values of λ stepwise by first searching for a t-design with a small λ, removing all orbits used for this design before the next step, and continuing this way until all orbits are covered by some t-design. Then we try to combine these designs into disjoint designs which all have the same parameter λ. Mathematically, this problem can be described by the solutions of a system of linear diophantine equations.

Theorem 2 *Let $t < k < v$ be natural numbers, and V a set of v points. Suppose that for natural numbers λ and N, $\lambda N = \binom{v-t}{k-t}$. Let P be a partition of $\binom{V}{k}$ into disjoint t-designs such that, for $j = 1, \cdots, n$, there are exactly a_j designs with parameter λ_j in P. Let $A = (a_{ij})$ be an $m \times n$ integer matrix such that $0 \leq a_{ij} \leq a_j$, and for each $i = 1, \ldots, m$:*

$$\sum_{j=1}^{n} a_{ij}\lambda_j = \lambda. \tag{3.1}$$

Then, each integer solution vector $(N_1, \cdots, N_m)$ to the diophantine system:

$$(N_1, \cdots, N_m)A = (a_1, \cdots, a_n) \tag{3.2}$$

determines a large set $LS[N](t, k, v)$ by selecting N_i $t-(v, k, \lambda)$ designs which correspond to the i^{th} row $(a_{i1}, \cdots, a_{in})$. In such a solution a_{ij} designs have parameters $t - (v, k, \lambda_j)$.

Proof: If $\mathcal{D}_1$ and $\mathcal{D}_2$ are two disjoint t-designs on the same point set V, with $\mathcal{D}_i$ a $t - (v, k, \ell_i)$ design, then their union $\mathcal{D}_1 \cup \mathcal{D}_2$ is a $t - (v, k, \ell)$ design with with $\ell = \ell_1 + \ell_2$. Each row of A allows us to construct a $t-(v, k, \lambda)$ design as the union of disjoint designs, and a solution $(N_1, \cdots, N_m)$ to (3.2) allows us to assemble exactly the correct number of disjoint $t - (v, k, \lambda)$ designs to cover $\binom{V}{k}$ exactly. □

Of course, there may be many different partition types of $\binom{V}{k}$ into disjoint t-designs, as well as different partitions of the same type. In general, Theorem 2 does not completely solve the problem of describing all $LS[N](t, k, v)$ which can be obtained from the orbits on $k-$subsets of a prescribed permutation group $G|V$.

There are some special cases where a finest partition of the set of all k-orbits into t-designs is unique. Obviously this is the case if the group has only one orbit on t-subsets, i. e. the group is t-homogeneous. Then, each $k-$orbit is a t-design and the finest partition is just the set of all k-orbits. So, in this case Theorem 2 allows us to find a complete solution.

As an example consider $G = P\Gamma L(2, 27)$ in its action on the projective line of $v = 28$ points. Let $k = 11$ and $t = 3$. Since this group is 3-homogeneous, each k-orbit is a 3-design. These orbits form our starting partition P. Here, there are $a_1 = 343$ designs with $\lambda_1 = 2970$, $a_2 = 33$ designs with $\lambda_2 = 1485$, and $a_3 = 14$ designs with $\lambda_3 = 990$. We note that 495 divides each of λ_1, λ_2 and λ_3. We have $\binom{v-t}{k-t} = \binom{25}{8} = 495 \cdot 5 \cdot 19 \cdot 23$ where each λ must be a multiple of 495. We search for an $LS[5](3, 11, 28)$ using Theorem 2. We can simplify our first equation

$$\sum_{j=1}^{n} a_{ij}\lambda_j = \lambda$$

by dividing both sides by 495 and get

$$6 \cdot a_{11} + 3 \cdot a_{12} + 2 \cdot a_{13} = 437 = 19 \cdot 23.$$

A first solution vector is $(a_{11}, a_{12}, a_{13}) = (72, 1, 1)$. A second solution is $(a_{21}, a_{22}, a_{23}) = (55, 29, 10)$. Then $(N_1, N_2) = (4, 1)$ solves diophantine system (3.2) so that we have to combine 4 designs of the first kind with 1 design of the second kind to get the desired $LS[5](3, 11, 28)$.

In what follows we present some further examples of large sets obtained by applying Theorem 2. We report here some orbit statistics obtained with the help of DISCRETA.

Examples 1

- An $LS[77](1, 6, 12)$. Here the cyclic group of order 12 regular on points, is acting on the set of all 6-subsets with 75 orbits of length 12, 3 orbits of length 6, 1 orbit of the length 4, and 1 of length 2.
- An $LS[11](2, 6, 13)$. Here the group $AGL(1, 13)$ has 9 orbits of size 156, 3 orbits of length 78, 1 orbit of length 52 and 1 of length 26. These can be combined into 11 1-designs with 156 blocks each. Using Alltop's theorem this large set extends to an $LS[11](3, 7, 14)$.
- An $LS[2](3, 6, 12)$. The group $PSL(2, 11)$ has 1 orbit of length 330, 2 orbits of length 132, and 3 orbits of length 110.
- An $LS[2](3, 8, 20)$. The group $PSL(2, 19)$ has 29 orbits of length 3420, 13 orbits of length 1710, 3 orbits of length 855, 3 orbits of length 570, and 1 orbit of length 285.
- An $LS[N](3, 11, 32)$. Here the group $PSL(2, 31)$ has 5 orbits of size $|PSL(2, 31)|/5$ and 12 orbits of length $|PSL(2, 31)|/3$. All other orbits have length $|PSL(2, 31)|$. Since in each case the number of orbits is a multiple of the stabilizer order, one can always combine orbits to form a design with exactly $|PSL(2, 31)|$ blocks. Thus, there exists an $LS[29 \cdot 13 \cdot 23](3, 11, 32)$.
- $G = P\Gamma L(2, 32)$ is 4-homogeneous on the 33 points of the projective line, and this group has 32 orbits on 7-sets. There are 22 orbits of length $|G|$, 7 orbits of length $|G|/2$ and 3 orbits of length $|G|/5$. Combining 4 orbits of length $|G|/2$ with two orbits of size $|G|$ gives $24 = 8 \cdot 3$ sets of size $|G|$. We combine 8 of these sets with one orbit of length $|G|/5$ three times and thus get an $LS[3](4, 7, 33)$.

- Again, let $G = P\Gamma L(2,32)$ act on 33 points as above. Then, in its induced action on 10-sets, G has 538 orbits of length $|G|$, 54 orbits of length $|G|/2$, 1 orbit of length $|G|/5$, and 3 orbits of length $|G|/10$. These can be combined to form an $LS[39](4,10,33)$, an $LS[29](4,10,33)$, an $LS[13](4,10,33)$, and an $LS[3](4,10,33)$.

- Consider $G = PGL(2,37)$ in its action on the projective line of 38 points. Then, in its induced action on 7-sets G has 225 orbits of length $|G|$, 46 orbits of length $|G|/2$, and 4 orbits of length $|G|/3$. These can be combined to an $LS[2](3,7,38)$.

In a special case we can determine N in a more convenient way.

Theorem 3 *If there exists a partition of $\binom{v}{k}$ into a_1 designs with parameters $t-(v,k,q\lambda)$ and a_2 designs with parameters $t-(v,k,\lambda)$ for some natural number q then large sets $LS[N](t,k,v)$ can be obtained from this partition for certain divisors N of $a_2 + qa_1$. For such an N there must exist a non-negative integer $m \leq a_2/q$ such that N divides both $a_2 - mq$ and $a_1 + m$.*

Proof: From the starting partition combine q of the $t-(v,k,\lambda)$ designs to form a $t-(v,k,q\lambda)$ design. If this is done m times there result $a_1 + m$ designs with parameters $t-(v,k,q\lambda)$. Suppose that $Nd = a_1 + m$ and $Ne = a_2 - mq$ for some natural numbers d, e. Combining d designs with parameters $t-(v,k,q\lambda)$ with e designs with parameters $t-(v,k,\lambda)$ results in a $t-(v,k,\lambda')$ design where $\lambda' = (dq+e)\lambda$. Repeating this N times until all designs are used results in the desired large set. The number N then must divide $a_2 + qa_1$. To see this simply insert $m = dN - a_1$ into $Ne = a_2 - mq$ to obtain $a_2 + a_1 q = (dq+e)N$. Thus, the possible values of N are divisors of $a_2 + qa_1$. □

The group $G = P\Gamma L(2,32)$ is 4-homogeneous on 33 points. The orbits of G partition the set of all 10-sets into 538 4-(33,10,840) designs, 54 4-(33,10,420) designs, 1 4-(33,10,168) design, and 3 4-(33,10,84) designs. The last three types give exactly 55 4-(33,10,420) designs. So, we get a partition into designs of two types as required for Theorem 3. We have $q = 2$, and $a_2 + 2a_1 = 1131 = 29 \cdot 39$. ¿From $55 = 8 \cdot 2 + 39$, and $538 + 8 = 546 = 39 \cdot 14$ we get an $LS[39](4,10,33)$. Also, from $55 = 13 \cdot 2 + 29$, and $538 + 13 = 551 = 29 \cdot 19$ we get an $LS[29](4,10,33)$. The divisors of 39 yield further large sets. But the divisors $3 \cdot 29$ and $13 \cdot 29$ are too large for a large set.

By the *translation* group $T(n,p)$ we mean a multiplicative elementary abelian group of order p^n. In the following theorem we consider the right regular action of $T(2,p)$ on itself.

Theorem 4 *Let p be a prime, and $G = V = T(2,p)$. Then there exists an $LS[p+1](1,p,p^2)$ where $T(2,p)$ acts as a group of automorphisms of each of the designs in the large set. Thus, the large set is $[G]-$invariant.*

Proof : The translation group $T(2,p)$ has exactly $p+1$ subgroups of order p, each leaving invariant under the group action of right multiplication the set of elements in that subgroup and its right cosets. No other subset of p elements is left invariant by such a subgroup. Each such collection of p-sets forms a single orbit under the full group. So, the whole group has exactly $p+1$ orbits of size p on p-sets. All other p-sets must lie in orbits of size p^2. There remain $\frac{1}{p^2}(\binom{p^2}{p} - p(p+1))$ orbits of size p^2. This number can be represented in the form $n \cdot (p+1)$ for some natural number n. So, we can form a large set of $p+1$ designs by composing each design out of n orbits of size p^2 and one orbit of size p. □

For example $p = 5$ yields 2124 orbits of size 25 and 6 orbits of size 5 on the 5-subsets of $T(2,5)$. Since 2124 is $354 \cdot 6$, a large set can be obtained by constructing disjoint designs each formed by combining 354 orbits of size 25 and one of size 5. If we had tried the cyclic group of order 25 we would have obtained only one orbit of size 5 and 2125 orbits of size 25. This would not produce any large sets. Thus, we see that no simple divisibility conditions can cover this case. One can easily generalize Theorem 4 to higher powers of p.

Generally, group orbits are a rich source for the required partitions. We will consider several series of groups and therefore look for special cases which are easier to verify in the context of Theorem 3. First we look for a case where $N = \binom{v}{k}/|G|$.

The case where all k-orbits have length the group order, i. e. semiregular large sets, has been treated theoretically for the groups $PSL(2,q)$ by Cusack and Magliveras [12]. We now consider a slightly more general case.

Theorem 5 *Let p be a prime and suppose that $G|V$ is a $t-$homogeneous group action where each orbit of G on $\binom{V}{k}$ has length either $|G|$ or $|G|/p$. Suppose further that $|G|$ divides $\binom{v}{k}$, $v = |V|$. Then, there exists an $LS[N](t,k,v)$, where $N = \binom{v}{k}/|G|$.*

Proof : Let G have a orbits of size $|G|$ and b orbits of size $|G|/p$. Then

$$a + b = z$$

is a natural number. The union of all orbits is the set of all k-subsets. Thus, we get a second equation

$$a\,|G| + b\,|G|/p = \binom{v}{k}.$$

Solving for b yields

$$b = \frac{p}{p-1}\{z - \frac{1}{|G|}\binom{v}{k}\}.$$

If $|G|$ divides $\binom{v}{k}$ then b is a multiple of p, so that we can repeatedly combine p orbits of size $|G|/p$ to form a t-design with $|G|$ blocks each until all orbits of this length are exhausted. Each of the remaining orbits of length $|G|$ also forms such a design. Since all orbits are disjoint, these designs form a partition of the complete design into designs with $|G|$ blocks each. Dividing $\binom{v}{k}$ by $|G|$ then gives the number of designs N. □

The cyclic group of order v, in its regular representation is of course transitive and thus, for $gcd(v,k) = 1$ or $gcd(v,k) = q$, where q is a prime and v divides $\binom{v}{k}$, we get an $LS[N](1,k,v)$, where $N = \binom{v}{k}/v$.

Several new large sets are obtained from Theorem 5 by looking at well known families of groups which are at least 2-homogeneous. In particular, we obtain the following result.

Theorem 6 *Let p be an odd prime. For $p \equiv 1$ mod 4 and f any natural number, or $p \equiv 3$ mod 4 and f an even natural number, let $d = p^f - 1$. For $p \equiv 3$ mod 4 and f an odd natural number let $d = (p^f - 1)/2$. Let $2 \leq k \leq v = p^f$ and suppose d divides $\binom{p^f-1}{k-1}$. Set $N = \frac{1}{d}\binom{p^f-1}{k-1}$. If p does not divide k and $gcd(k(k-1), d) \in \{1, q\}$ for some prime q then there exists an $LS[N](2,k,v)$.*

Proof : If $p \equiv 1$ mod 4, f arbitrary; or $p \equiv 3$ mod 4 and f even, then $AGL(1,p^f)$ is 2-homogeneous on p^f points. If $p \equiv 3$ mod 4 and f is odd, then $AGL(1,p^f)$ and its unique subgroup $AGL(1,p^f)/2$ of index 2 are 2-homogeneous on p^f points.

In order to apply Theorem 5 we have to find out for which values of k such a group G has orbits of sizes $|G|$ or $|G|/q$ only, where q is some prime. So, we examine the cycle types of the elements of G to see whether a k-subset is left invariant. We have to single out the cases where, besides the identity, either no elements or only elements of some fixed prime order q may leave such a k-subset invariant.

The group $AGL(1,p^f)$ has an elementary abelian normal subgroup of order p^f and one conjugacy class of complements which are cyclic of order $p^f - 1$. Elements other than the identity have either order p, comprised of cycles of length p only, or have just one fixed point and $(p^f-1)/d$ cycles of length d for some divisor d of p^f-1. The subgroup $AGL(1,p^f)/2$ contains the elements of order p and those further elements whose order divides $(p^f-1)/2$.

If $G = AGL(1,p^f)$ then for k prime to p and $gcd(k(k-1),p^f-1) \in \{1,q\}$ for some prime q, the condition that $|AGL(1,p^f)| = p^f(p^f-1)$ divides $\binom{p^f}{k}$ suffices to obtain an $LS[N](2,k,p^f)$.

If $p \equiv 3 \bmod 4$ and f is odd then for k prime to p and $gcd(k(k-1),(p^f-1)/2) \in \{1,q\}$ for some prime q, the condition that $|AGL(1,p^f)| = p^f(p^f-1)/2$ divides $\binom{p^f}{k}$ suffices to obtain an $LS[N](2,k,p^f)$. $\square$

Examples 2

- $G = AGL(1,17)$ has order $17 \cdot 16$. For $k=6$ we have 42 orbits of size 272 and 7 orbits of size 15. So there is an $LS[7](2,6,17)$. $G = AGL(1,23)/2$ has order $23{\cdot}11$. For $k=5$ we have 133 orbits of size 253. So there is an $LS[133](2,5,23)$. For $G = AGL(1,27)$ we find an $LS[50](2,4,27)$, an $LS[230](2,5,27)$, and an $LS[2530](2,7,27)$.

- $G = AGL(1,29)$ has 12298 orbits of size $|G|$, 70 orbits of size $|G|/2$, and 3 orbits of size $|G|/4$ on 9-sets. Combining twice 2 of the orbits of size $|G|/2$ adds two more designs with $|G|$ blocks to the regular ones. Thus, there are now 12300 of these designs, 66 designs with $|G|/2$ blocks, and the remaining 3 designs with $|G|/4$ blocks. Each of these numbers is a multiple of 3 so, one can build an $LS[3](2,9,29)$ out of this orbit partition.

- $G = AGL(1,31)$ has 646305 orbits of size $|G|$, 50 orbits of size $|G|/3$, 6 orbits of size $|G|/5$, and 2 orbits of size $|G|/15$ on 15-sets. The corresponding values of λ are 105, 35, 21, 7 respectively. Combining the two designs of $\lambda = 7$ with 1 design of $\lambda = 21$ yields one more design with $\lambda = 35$. The remaining 5 designs with $\lambda = 21$ yield one further design with $\lambda = 105$. Now, both numbers 646306 and 51 are divisible by 17 so that we get an $LS[17](2,15,31)$. By usings Alltop's construction, see below, we get an $LS[17](3,16,32)$.

Theorem 7 *Let p be an odd prime. For $p \equiv 1 \bmod 4$ and f any natural number, or $p \equiv 3 \bmod 4$ and f an even natural number let $d = p^f - 1$. For $p \equiv 3 \bmod 4$ and f an odd natural number let $d = (p^f-1)/2$. Let $2 \le k \le v = p^f$. Suppose that d divides $\binom{p^f-1}{k-1}$ and let $N = \frac{1}{d}\binom{p^f-1}{k-1}$. If $gcd(p,k(k-1) = 1$ and at most one of the greatest common divisors $gcd(k(k-1)(k-2),d)$, $gcd(k,p^f+1)$ (respectively*

$gcd(k,(p^f+1)/2$ when $p^f \equiv 3 \bmod 4$), is a prime q and otherwise the gcd is always 1, then there exists an $LS[N](3,k,v)$.

Proof : $PGL(2,p^f)$ is 3-homogeneous on p^f+1 points. If $p \equiv 3 \bmod 4$ and f is an odd number then already the unique subgroup $PSL(2,p^f)$ of index 2 in $PGL(2,p^f)$ is 3-homogeneous on p^f+1 points.

The group $PGL(2,p^f)$ has degree p^f+1 and any non-identity element fixes at most 2 points. There are elements with no fixed points, i. e. powers of a cycle of length p^f+1. There are elements with exactly one fixed point, i. e. elements of order p. Finally, there are elements with exactly 2 fixed points, these are the powers of elements of order p^f-1.

In the case of a 3-homogeneous $PSL(2,p^f)$ we have to restrict these elements to elements of order p, the powers of elements of order $(p^f+1)/2$, and the powers of elements of order $(p^f-1)/2$.

So, if no k-subset is fixed by a non-identity element then k must be prime to p^f+1 or $(p^f+1)/2$, respectively, $k(k-1)$ must be prime to p^f and $k(k-1)(k-2)$ must be prime to p^f-1 or $(p^f-1)/2$, respectively. These conditions are carefully examined by Cusack and Magliveras in the case of a 3-homogeneous $PSL(2,p^f)$.

Now we want to allow that there may exist orbits of length $|PSL(2,p^f)|/q$ for a fixed prime $q>2$. Then one of the coprime conditions may be replaced by a common divisor q. Thus we get the following cases.

- $gcd(k,(p^f+1)/2)=q$, $gcd(k(k-1),p^f)=1$, and $gcd(k(k-1)(k-2),(p^f-1)/2)=1$.
- $gcd(k,(p^f+1)/2)=1$, $gcd(k(k-1),p^f)=1$, and $gcd(k(k-1)(k-2),(p^f-1)/2)=q$.

□

Consider the case $p^f \equiv 1 \bmod 4$. By [12] for any group containing $PSL(2,p^f)$ there exists no semiregular large set with $t=3$. Since $PSL(2,p^f)$ is no longer 3-homogeneous, we proceed to use $G=PGL(2,p^f)$. We have to drop semiregularity and still obtain restrictions for possible large sets. It is easy to see that $gcd(p^f-1,k(k-1)(k-2))=1$ is impossible for odd p. If $gcd(p^f-1,k(k-1)(k-2))=q$ for a prime q then $q=2$ and $k \equiv 3 \bmod 4$. There are examples for $p^f=53$, $k=7$ and $p^f=29$, $k=11$ below.

For $p^f=32$ there exists an $LS[5](3,4,33)$ with given automorphism group $PGL(2,32)$. The case $p^f=64$ does not produce a large set. See also Teirlinck [25].

Examples 3

- $G = PSL(2,23)$ is 3-homogeneous on 24 points and its order divides $\binom{24}{10}$. So there exists an $LS[323](3,10,24)$. We remark that $LS[7](3,5,24)$, $LS[57](3,7,24)$, and $LS[412](3,11,24)$ also exist by [13].
- $G = PSL(2,31)$ is 3-homogeneous on 32 points and for $k = 9$ and $p = 3$ the conditions of the Theorem hold. So, there exists an $LS[1885](3,9,32)$. Since $1885 = 5 \cdot 13 \cdot 29$, we also get parameter sets $LS[5](3,9,32)$, $LS[13](3,9,32)$, and $LS[29](3,9,32)$.
- $G = PGL(2,53)$ is 3-homogeneous on 54 points and its order divides $\binom{54}{7}$. Now, 7 is coprime to 54 and $7 \cdot 6 \cdot 5$ has greatest common divisor 2 with 52. In fact there are 1140 orbits of length $|G|$ and 100 orbits of length $|G|/2$. So, there exist $LS[N](3,7,54)$ for $N = 2,5,7,17$.

Theorem 3 gives rise to large sets in more general situations than Theorem 4. However, it is more difficult to derive easy conditions which are sufficient for Theorem 3. The strategy followed after Theorem 4, to consider restricted families of groups, will also be followed here. So, we look again at t-homogeneous groups and determine the distribution of orbits of k-sets with particular orbit lengths in convenient special cases.

Let G act t-homogeneously on a set V. Then the condensed version of a Kramer-Mesner matrix needed for Theorem 2 for t- versus k-orbits can be computed by combinatorial methods. The idea is to classify each k-orbit by the conjugacy class of a corresponding stabilizer subgroup. If the stabilizer orders are known Alltop's Lemma [4], [7] yields the matrix entries.

The first step is to determine for each conjugacy class of subgroups and for any particular representative U in the conjugacy class, the k-sets invariant under U. Clearly, such a k-set K must consist of full U-orbits. So,

$$K = K_{11} \cup \cdots \cup K_{1k_1} \cup \cdots \cup \; K_{i1} \cup \cdots \cup K_{ik_i} \cdots \cup K_{n1} \cup \cdots \cup K_{nk_n}$$

with k_i orbits K_{ij} of size i up to some n. If U has exactly a_i orbits of size i then there are
$\binom{a_i}{k_i}$ possibilities to select k_i orbits of size i. For a given pattern $(k_1, \cdots, k_n)$ with $k = \sum_i i \cdot k_i$ the possibilities for the different sizes i are multiplied to obtain the number of combinations. This gives the general formula

$$C_{\binom{V}{k}}(U) \;=\; \sum_{k=\sum_i i \cdot k_i} \prod_i \binom{a_i}{k_i}$$

for the number of k-sets invariant under U.

For any subgroup U of G the orbits of U are in bijection to the double cosets $S\backslash G/U$ of the stabilizer S of a point, and U in G. These again can be classified according to the stabilizers of right cosets Sg in U with respect to right multiplication by elements of U, see [20]. For a subgroup D of U we have $SgD = Sg$ if and only if gDg^{-1} is contained in S.

In the case of $G = PSL(2,p)$ for a prime $p \equiv 3 \bmod 4$ the stabilizer of a point is a semidirect product of $\mathbb{Z}_p$ by $\mathbb{Z}_\ell$ where $\ell = (p-1)/2$ is odd. So, finding the orbits of a subgroup U of $PSL(2,p)$ of which the order is not divisible by p can be done by deciding which subgroups of U may be conjugate to a subgroup of S. Such a subgroup D is cyclic of an order dividing ℓ. In particular $|D|$ is odd and any subgroup of some order 2^f has only orbits of length 2^f. The subgroups of $PSL(2,p)$ are well known, see [8]. Those which may have a nontrivial intersection with S and of which the order is not divisible by p must have an intersection which is either $\mathbb{Z}_3$ or $\mathbb{Z}_5$. So, it is easy to handle these few cases. Then for each U the orbits can be written down.

To obtain the number of k-sets where U is the full stabilizer apply the principle of inclusion and exclusion to the these numbers over all overgroups of U. This is equivalent to summing over these numbers weighted by the values of the Moebius function. Again double counting allows us to determine the number of overgroups H containing U, and belonging to a certain conjugacy class, from the number of conjugates of U that are contained in some H. So, if the subgroups are well known as in the case of $PSL(2,p)$ this Moebius inversion can be carried out, see also [14], [22].

Because elements with the same stabilizer U generally lie in the same orbit only if they are already in the same orbit under the normalizer $N_G(U)$, the number obtained has to be divided by the orbit length $|N_G(U)/U|$ which all $N_G(U)$ orbits have on this set.

As an example consider orbits of 4-sets under $PSL(2,23)$ acting on 24 points. There is just one conjugacy class of subgroups $\mathbb{Z}_2$. Such a subgroup U is generated by an involution fixing no points, so it must have point-orbits of length 2 only. Hence, there are $\binom{12}{2} = 66$ $U-$invariant 4-sets. A cyclic group $\mathbb{Z}_4$ is generated by an element of type 4^6 and therefore must have point-orbits of length 4. The Sylow 2-subgroups are dihedral and thus contain a unique cyclic subgroup of order 4. Thus, such a subgroup has $\binom{6}{1} = 6$ invariant 4-sets. All these lie in the same orbit of the normalizer of this cyclic subgroup which is dihedral of order 24. There remain two conjugacy classes of Klein subgroups V_4 which also have orbits of length

4 only. Each has as its normalizer a subgroup isomorphic to S_4 which again forms one orbit out of the 4-sets invariant under a V_4. No larger group has orbits of length 4. A group of order 2 then is contained in a unique cyclic subgroup of order 4 and in 3 subgroups from each conjugacy class of V_4's by double counting. So, for each of these 7 subgroups of order 4 containing the $\mathbb{Z}_2$, we have to subtract from the 66 invariant 4-sets their 6 invariant 4-sets. The remaining 24 invariant 4-sets fall into 2 orbits of length 12 under the normalizer of $\mathbb{Z}_2$ which is dihedral of order 24.

If we restrict the possible values of k by applying the appropriate divisibility conditions, then only certain subgroups can occur as stabilizers of a k-set. So, if all such stabilizers have to be contained in a certain cyclic subgroup of order h up to conjugacy, then we can use the number theoretic Moebius function for the lattice of divisors of h. We use this strategy below to obtain large sets where the number of k-orbits is fairly large.

To obtain the values of λ in these cases by Alltop's Lemma we recall the formula

$$|N_G(K)| \cdot m(T, K^G) = |N_G(T)| \cdot m'(T^G, K), \tag{3.3}$$

where $N_G(R)$ denotes the stabilizer in G of subset $R \subset V$, $m(T, K^G)$ denotes the number of k-sets in the orbit K^G that contain the t-set T and $m'(T^G, K)$ denotes the number of t-sets in the orbit T^G that are contained in the k-set K. Here we assume that G is t-homogeneous so that $m'(T^G, K) = \binom{k}{t}$. The value of λ is just $m(T, K^G)$ and can be obtained from the formula when the stabilizer orders are known.

Let $G = AGL(1, p^f)/2$ be the unique subgroup of index 2 in $AGL(1, p^f)$ for some prime power $p^f \equiv 3 \bmod 4$. Then G acts regularly on 2-sets. So, $|N_G(T)| = 1$ and if K is any k-set and T a 2-set we get that T is contained in exactly $\binom{k}{2}/|N_G(K)|$ of the k-sets in the orbit of K.

We consider special cases. So, we first assume that p does not divide k. Then no $k-$subset can be left invariant by an element of order p in G. So, the stabilizer of such a K must lie in one of the complements, C say. Since G is primitive and solvable, these complements form one conjugacy class and are maximal subgroups. In particular they are the normalizers of each of their non-identity subgroups. Any element of G mapping a k-set with stabilizer U onto another k-set with stabilizer U must normalize that stabilizer. Thus, if $U < C$ is the set-stabilizer of a $k-$subsets, then G has $a/[C : U]$ orbits of this type.

Any subgroup U of order d of C has one fixed point and $\frac{p^f-1}{2d}$ orbits of size d.

A fixed k-set must be a union of U-orbits. So, if d divides k there are exactly

$$\binom{\frac{p^f-1}{2d}}{\frac{k}{d}}$$

invariant k-subsets. If d divides $k-1$ there are exactly

$$\binom{\frac{p^f-1}{2d}}{\frac{k-1}{d}}$$

invariant k-subsets. In all other cases there are no invariant k-subsets.

This discussion leads to the following result. If p does not divide k and $gcd(k,(p^f-1)/2)=q$ for some prime q then G has exactly

$$\frac{2q}{p^f-1}\binom{\frac{p^f-1}{2d}}{\frac{k}{d}}$$

orbits on k-sets with stabilizer of order q. If p does not divide k and $gcd(k-1,(p^f-1)/2)=q$ for some prime q then G has exactly

$$\frac{2q}{p^f-1}\binom{\frac{p^f-1}{2d}}{\frac{k-1}{d}}$$

orbits on k-sets with stabilizer of order q. All other k-sets have trivial stabilizers.

Consider the case of $PGL(2,29)$. This group is 3-homogeneous and for $k=11$, besides the identity only an element of order 2 may fix such a k-set. The group order does not divide $\binom{30}{11}$, hence, Theorem 4 is not applicable. But Theorem 3 may be applied. ¿From the formulas we obtain 2171 designs with $\lambda=990$ and 495 designs with $\lambda=495$. So, $q=2$. In the context of Theorem 3 we have to find solutions for the system $143 \equiv 2m \bmod N$ and $2171 \equiv -m \bmod N$. So, N must divide $4485 = 3\cdot 5\cdot 13\cdot 23$. We get an $LS[N](3,11,30)$ for $N=3,5,13,23$.

Instead of constructing a large set from disjoint designs Magliveras [13] has also searched for all designs with the required λ which can be combined from the orbits of a prescribed group and then selected a partition into disjoint designs among those. As in Theorem 2, this strategy splits the general problem into two subproblems. First, all solutions of the Kramer-Mesner system of diophantine equations for the appropriate value of λ are computed and secondly a matrix is formed by the solution vectors of the system which again defines a diophantine system of linear equations with right hand side constant to 1. A 0-1 solution vector determines a large set by selecting those designs which correspond to a 1. Thus, this strategy uses the same tool in both steps. In many cases however this approach needs too much computer time or space. So, as in the strategy of Theorem 2 we try to directly construct a partition of $\binom{V}{k}$ into designs all with the same λ. This corresponds to backtracking in the second part of Magliveras strategy but avoiding to find all solutions in the first part. Because we may still have to run through a large solution space we have implemented a random selection of designs disjoint from those selected previously in the search for a partition into designs. Both versions are part of DISCRETA and led to some interesting new large sets with $t \geq 4$:

- $LS[3](4,6,13)$ with automorphism group $\mathbb{Z}_{13}$,
- $LS[3](5,7,24)$ with automorphism group $PGL(2,23)$,
- $LS[3](3,7,21)$ with automorphism group $PSL(3,4)$.
- $LS[5](2,3,17)$ admitting $\mathbb{Z}_{17}$
- $LS[29](3,5,32)$ with automorphism group $AGL(1,32)$.
- $LS[7](4,5,32)$ with automorphism group $AGL(1,32)$.

Note that by Lu[23], [27] large sets $LS[N](2,3,v)$ exist for $v \equiv 1$ or $3(mod\ 6)$ and $v \neq 7$. Here we have a large set for $v = 17$.

4 Recursive Constructions

Let p be an odd prime. Since for $0 \leq k \leq n < p$ no $\binom{n}{k}$ is divisible by p, Pascal's triangle mod p has non-zero entries in the first p rows. We call the triangle formed by the first p rows the *starting triangle.* The $(p+1)^{st}$ row then has 1's in the first and last entry and all other entries 0. So, we have the two ones as starting points of new triangles of p rows which are identical to the starting triangle. The entries outside these triangles in these p rows are 0. In the next $(2p+1)^{st}$ row the middle entry is 2, since the 1's from the two upper neighbors add up to 2. Apart from the border entries of 1's all other entries are 0. The 2 gives rise to a triangle of p rows in which each entry of the starting triangle with p rows is multiplied by 2 mod p. Generally, we obtain the following pattern: The Pascal triangle mod p is formed of triangles of p rows which are obtained from the starting triangle by multiplying it by some factor mod p. These factors themselves again form a Pascal triangle of the same type. So, a recursive structure is generated which easily describes the whole pattern. As an example, we display below Pascal's triangle modulo 5.

Pascal's Triangle mod 5

```
0   1
1   1 1
2   1 2 1
3   1 3 3 1
4   1 4 1 4 1
5   1 0 0 0 0 1
6   1 1 0 0 0 1 1
7   1 2 1 0 0 1 2 1
8   1 3 3 1 0 1 3 3 1
9   1 4 1 4 1 1 4 1 4 1
10  1 0 0 0 0 2 0 0 0 0 1
11  1 1 0 0 0 2 2 0 0 0 1 1
12  1 2 1 0 0 2 4 2 0 0 1 2 1
13  1 3 3 1 0 2 1 1 2 0 1 3 3 1
14  1 4 1 4 1 2 3 2 3 2 1 4 1 4 1
15  1 0 0 0 0 3 0 0 0 0 3 0 0 0 0 1
16  1 1 0 0 0 3 3 0 0 0 3 3 0 0 0 1 1
17  1 2 1 0 0 3 1 3 0 0 3 1 3 0 0 1 2 1
18  1 3 3 1 0 3 4 4 3 0 3 4 4 3 0 1 3 3 1
19  1 4 1 4 1 3 2 3 2 3 3 2 3 2 3 1 4 1 4 1
20  1 0 0 0 0 4 0 0 0 0 1 0 0 0 0 4 0 0 0 0 1
21  1 1 0 0 0 4 4 0 0 0 1 1 0 0 0 4 4 0 0 0 1 1
22  1 2 1 0 0 4 3 4 0 0 1 2 1 0 0 4 3 4 0 0 1 2 1
23  1 3 3 1 0 4 2 2 4 0 1 3 3 1 0 4 2 2 4 0 1 3 3 1
24  1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1
25  1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
26  1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
27  1 2 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 1
28  1 3 3 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 3 3 1
29  1 4 1 4 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 4 1 4 1
30  1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1
31  1 1 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 1
32  1 2 1 0 0 1 2 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 1 0 0 1 2 1
33  1 3 3 1 0 1 3 3 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 3 3 1 0 1 3 3 1
34  1 4 1 4 1 1 4 1 4 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 4 1 4 1 1 4 1 4 1
```

As we saw in section 2, a necessary condition for an $LS[N](t,k,v)$ to exist is that the component designs have λ such that $\lambda N = \binom{v-t}{k-t}$. Since a t-design is also an s-design for $0 \leq s \leq t$ and the large set an $LS[N](s,k,v)$, the λ_s-values also satisfy the equations $\lambda_s N = \binom{v-s}{k-s}$. Thus, for $s = 0,\ldots,t$, each of the binomial coefficients $\binom{v-s}{k-s}$ must be divisible by N. For $N = p$ a prime, Pascal's triangle mod p, as described above, easily allows us to locate the *admissible* parameter sets. For a given v and k, the $t+1$ entries parallel to the right border and ending with the position of $\binom{v-t}{k-t}$ mod p must be 0. So, one finds out that certain lower segments of the observed "inverted" 0-triangles with a base line of 0-entries, form the admissible parameter sets.

LS[3](3,k,v)

8																				-	-	-	-	-
9																				-	-	-	-	-
10																			-	-	-	-	-	-
11																			-	-	-	-	-	-
12																		-	-	-	-	?	?	?
13																		-	-	-	-	-	?	?
14																	-	-	-	-	-	-	?	?
15																	-	-	-	-	-	-	-	7
16																-	-	-	-	-	-	-	-	?
17																-	-	-	-	-	-	-	-	-
18															-	-	-	-	-	-	-	-	-	-
19															-	-	-	-	-	-	-	-	-	-
20														-	-	-	-	-	-	-	-	-	-	-
21														-	-	-	-	?	?	?	?	?	-	-
22													-	-	-	-	-	?	?	?	?	-	-	-
23													-	-	-	-	-	-	?	?	?	-	-	-
24												-	-	-	-	-	-	-	?	?	-	-	-	-
25												-	-	-	-	-	-	-	-	?	-	-	-	-
26											-	-	-	-	-	-	-	-	-	-	-	-	-	-
27											-	-	-	-	-	-	-	-	-	-	-	-	-	-
28										-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
29										-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
30									-	-	-	-	?	?	?	?	?	?	?	?	?	?	?	?
31									-	-	-	-	-	?	?	?	?	?	?	?	?	?	?	?
32								-	-	-	-	-	-	?	?	?	?	?	?	?	?	?	?	?
33								-	-	-	-	-	-	-	?	?	?	?	?	?	?	?	?	?
34							-	-	-	-	-	-	-	-	?	?	?	?	?	?	?	?	?	?
35							-	-	-	-	-	-	-	-	-	?	?	?	?	?	?	?	?	?
36						-	-	-	-	-	-	-	-	-	-	?	?	?	?	?	?	?	?	?
37						-	-	-	-	-	-	-	-	-	-	-	?	?	?	?	?	?	?	?
38					-	-	-	-	-	-	-	-	-	-	-	-	?	?	?	?	?	?	?	?
39					-	-	-	-	?	?	?	?	?	-	-	-	-	?	?	?	?	?	?	?
40				-	-	-	-	-	?	?	?	?	-	-	-	-	-	?	?	?	?	?	?	?
41				-	-	-	-	-	-	?	?	?	-	-	-	-	-	-	?	?	?	?	?	?
42			-	-	-	-	-	-	-	?	?	-	-	-	-	-	-	-	?	?	?	?	?	?
43			-	-	-	-	-	-	-	-	?	-	-	-	-	-	-	-	-	?	?	?	?	?
44		-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	?	?	?	?	?
45		-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	?	?	?	?
46	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	?	?	?	?
47		-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	?	?	?
48	-	-	-	?	?	?	?	?	-	-	-	-	?	?	?	?	?	-	-	-	-	?	?	?
49		-	-	-	?	?	?	?	-	-	-	-	-	?	?	?	?	-	-	-	-	-	?	?
50	-	-	-	-	?	?	?	-	-	-	-	-	-	?	?	?	-	-	-	-	-	-	?	?
51		-	-	-	-	?	?	-	-	-	-	-	-	-	?	?	-	-	-	-	-	-	-	?
52	-	-	-	-	-	?	-	-	-	-	-	-	-	-	?	-	-	-	-	-	-	-	-	?
53		-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
54	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
55		-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
56	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
57		?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	-	-
58	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	-	-	-
59		?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	-	-	-
60	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	-	-	-	-
61		?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	-	-	-	-

We display above an example of the admissible parameters for $LS[3](3,k,v)$ where, because of symmetry we only show the part of the table with $k \leq v/2$. In this display, we note that there are 4 diagonals, corresponding to $k \leq 3$, for which there are no admissible parameters. In later tables we drop such empty diagonals, and no confusion will occur because feasible parameters where existence is known are labeled by the corresponding value of k. Such a k-label, then, identifies the corresponding diagonal of the table.

For example, when $N = 3$ and $t = 3$ we get a triangle of possible parameters of which we only show the part for $k \leq v/2$. Each row is indexed by v. The positions in the triangle correspond to the positions in Pascal's triangle. A question mark indicates a value of k for which an $LS[3](3, k, v)$ is admissible from the divisibility conditions. We replace question marks by the value of k when a large set with the corresponding parameter set is known to exist and by a $-$ sign if it is known that such a large set cannot exist. The question marks that remain identify the still undecided cases. The same kind of tables have already been used for halvings in [21].

If each of the base line parameter sets of an admissible triangle belongs to an existing large set then the whole triangle does so. This follows from one of the theorems cited below. There are further recursion rules allowing us to fill some triangle for larger values of v if a triangle for smaller values can be filled with an existence sign.

Lemma 1 *If all designs in an $LS[N](t, k, v)$ are derived with respect to the same point x, then the resulting designs form an $LS[N](t-1, k-1, v-1)$, furthermore, the corresponding residual designs with respect to x form an $LS[N](t-1, k, v-1)$.*

Lemma 2 *Alltop's Construction: If an $LS[N](2s, k, 2k+1)$ exists then also an $LS[N](2s+1, k+1, 2k+2)$ exists.*

This follows from applying Alltop's construction [5] to each design in the given large set.

Theorem 8 *Ajoodani-Namini[2]: If an $LS[p](t, k, v)$ exists, p a prime, then there also exist $LS[p](t+1, pk+j, p(v+1))$ for $0 < j < p$.*

Corollary 1 *If an $LS[p](t, k, v)$ exists, p a prime, then there also exist $LS[p](t, pk+j, p(v+1)-1)$ for $j = 0, 1, \ldots, p-1$.*

To see this, form the derived and residual large sets from the large sets resulting from Ajoodani-Namini's theorem.

Theorem 9 *If an* $LS[3](2s, k, 2k+1)$ *exists, then an infinite series of large sets* $LS[3](2(s+i), (k+2)3^i - 2, (2k+4)3^i - 3)$ *exists, where* $i = 0, 1, \ldots$.

Proof: First apply Alltop's construction to the given $LS[3](2s, k, 2k+1)$ to obtain an $LS[3](2s+1, k+1, 2k+2)$. ¿From this, by the corollary to Ajoodani-Namini's Theorem, construct an $LS[3](2s+2, 3k+4, 6k+9)$. This again fulfills the assumption of the theorem. Now the formula follows by induction. □

The new $LS[3](4, 6, 13)$ is a starting point of the series $LS[3](2(2+i), 6\cdot 3^i - 2, 16\cdot 3^i - 2)$ for $i = 0, 1, \ldots$. This series has the smallest presently known values of v for a t-(v, k, λ) design with a given $t \geq 16$. An $LS[3](4, 5, 13)$ had been found earlier by [17]. The supplementary designs form an $LS[3](4, 7, 13)$ and an $LS[3](4, 8, 13)$ respectively. From these large sets then, the next Theorem yields large sets $LS[3](4, 6, 14)$, $LS[3](4, 7, 14)$, $LS[3](4, 8, 14)$, $LS[3](4, 7, 15)$, $LS[3](4, 8, 15)$, and $LS[3](4, 8, 16)$. In particular, the $LS[3](4, 7, 15)$ is again a starting point for an infinite series obtained by applying Theorem 9. Further series can be obtained in a similar fashion from the large sets $LS[3](2, 5, 11)$, $LS[3](2, 6, 13)$, and $LS[3](2, 7, 15)$, but for these the values of v for a given t are larger than what we get from the first series.

Theorem 10 *Ajoodani-Namini, Khosrovshahi[16]: If an* $LS[N](t, k, v)$ *and an* $LS[N](t, k+1, v)$ *exist then so does an* $LS[N](t, k+1, v+1)$.

Theorem 11 *Ajoodani-Namini, Khosrovshahi[16]: If there exist* $LS[N](t, k, v)$ *for* $k = t+1, \ldots, \ell$ *and* $LS[N](t, k, u)$ *for all* k *in an interval* $a \leq k \leq \ell$ *then there exist* $LS[N](t, k, v+u-t)$ *for all* k *in the interval* $a \leq k \leq \ell$.

5 Tables

¿From the basic large sets, the above recursive results yield a large number of parameter sets, indicated compactly by entry k in row v of a table. As an example, we exhibit the table for $LS[3](3, k, v)$ below, and all other tables encompassing all that is known for $v \leq 61$ appear in the Appendix. An entry "-" means that the parameter set is not feasible, a question mark "?" means that the existence of a large set with that parameter set is undecided, and bold face indicates a new parameter set.

LS[3](3,k,v)

8																											-
9																											-
10																										-	-
11																										-	-
12																									4	5	6
13																									-	5	6
14																								-	-	6	7
15																								-	-	-	7
16																							-	-	-	-	8
17																							-	-	-	-	-
18																						-	-	-	-	-	-
19																						-	-	-	-	-	-
20																					-	-	-	-	-	-	-
21																					4	5	6	**7**	8	-	-
22																				-	5	6	7	8	-	-	-
23																				-	-	6	7	8	-	-	-
24																			-	-	-	7	8	-	-	-	-
25																			-	-	-	-	8	-	-	-	-
26																		-	-	-	-	-	-	-	-	-	-
27																		-	-	-	-	-	-	-	-	-	-
28																	-	-	-	-	-	-	-	-	-	-	-
29																	-	-	-	-	-	-	-	-	-	-	-
30																4	5	6	**7**	8	?	?	**11**	?	?	?	?
31																-	5	6	7	8	?	?	?	?	?	?	?
32															-	-	6	7	8	**9**	**10**	?	?	**13**	?	?	?
33															-	-	-	**7**	8	**9**	**10**	?	?	?	?	?	?
34														-	-	-	-	8	**9**	**10**	?	?	?	?	?	?	?
35														-	-	-	-	-	9	10	11	**12**	**13**	**14**	15	16	17
36													-	-	-	-	-	-	10	11	**12**	**13**	**14**	**15**	16	17	18
37													-	-	-	-	-	-	-	11	12	13	14	15	16	17	18
38												-	-	-	-	-	-	-	-	12	13	14	15	16	17	18	19
39												4	5	6	**7**	8	-	-	-	-	13	14	15	16	17	18	19
40											-	5	6	7	8	-	-	-	-	-	14	15	16	17	18	19	20
41											-	-	6	7	8	-	-	-	-	-	-	**15**	**16**	**17**	**18**	**19**	**20**
42										-	-	-	**7**	8	-	-	-	-	-	-	-	**16**	**17**	**18**	**19**	**20**	**21**
43										-	-	-	-	8	-	-	-	-	-	-	-	-	17	18	19	20	21
44									-	-	-	-	-	-	-	-	-	-	-	-	-	-	**18**	**19**	**20**	**21**	**22**
45									-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**19**	**20**	**21**	**22**
46								-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	20	21	22	23
47								-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	**21**	**22**	**23**
48							4	5	6	7	8	-	-	-	-	13	?	**15**	?	17	-	-	-	-	**22**	**23**	**24**
49							-	5	6	7	8	-	-	-	-	-	?	?	?	?	-	-	-	-	-	23	24
50						-	-	6	7	8	-	-	-	-	-	-	?	?	?	-	-	-	-	-	-	**24**	**25**
51						-	-	-	**7**	8	-	-	-	-	-	-	-	?	?	-	-	-	-	-	-	-	**25**
52					-	-	-	-	8	-	-	-	-	-	-	-	-	?	-	-	-	-	-	-	-	-	**26**
53					-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
54				-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
55				-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
56			-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
57			4	5	6	7	8	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	-	-
58		-	5	6	7	8	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	-	-	-
59		-	-	6	7	8	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	-	-	-
60	-	-	-	7	8	**9**	?	11	?	13	**14**	?	?	17	?	19	?	**21**	**22**	23	?	**25**	?	-	-	-	-
61		-	-	-	8	**9**	?	?	?	?	**14**	?	?	?	?	?	?	?	**22**	**23**	?	?	?	-	-	-	-

The Appendix containing all the tables generated for this paper can be downloaded from one of the web sites: `<http://helios.unl.edu/~spyros/>` or `<http://www.mathe2.uni-bayreuth.de/people/laue.html>`. We hope the reader will be interested in resolving some of the unsettled cases.

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