Genealogy of t-designs

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Abstract

Recent years saw a dramatic increase in existence results for t-designs with large t, i.e. $t \geq 5$. Designs are now known to exist for several thousand parameter sets, mostly constructed by the method of orbiting under a group. This note is a contribution to the classification of these designs by parameters. We take an abstract look at admissible parameter sets in general. We introduce a partial order, reflecting relationships between designs, and we analyze the structure of the resulting posets. The parameter sets of known designs fall in no more than 100 categories, which we call ancestor clans.

Keywords: t-design, parameter set, ancestor, family, clan.

1 Introduction

Let t, v, k and λ be natural numbers. A t-(v, k, λ) design is a pair $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a set of v elements called points and $\mathcal{B} = \{B_1, \ldots, B_b\}$ is a set of k-subsets of $\mathcal{V}$ – called blocks – such that every t-element subset of $\mathcal{V}$ is contained in exactly λ blocks. The quadruple of integers t-(v, k, λ) is called the parameter set of the design. The integer t is the *point regularity*, v is the *size of the underlying point set*, k is the *block size* and λ is the *index*. The *number of blocks*, b, is determined by t, v, k and λ as $b = \lambda\binom{v}{t}/\binom{k}{t}$. A design with $\lambda = 1$ is called *Steiner System.* Certain designs are so obvious that one considers them as trivial. One of these is the *complete design* which consists of *all* k-subsets. It is a t-design for all $t \leq k$. The parameters as a k-design are k-$(v, k, 1)$, with $b = \binom{v}{k}$. Let us recall some more parameters of t-designs. For nonnegative integers i and j with $i + j \leq t$, and for I and J fixed disjoint subsets of points of size i and j, respectively, the number of blocks containing I and disjoint from J is a constant, denoted as $\lambda_{i,j}$. Ray-Chaudhuri and Wilson [6] proved that

$$\lambda_{i,j} = \lambda \frac{\binom{v-i-j}{k-i}}{\binom{v-t}{k-t}} \quad \text{for } i + j \leq t. \tag{1}$$

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We consider the following relationships between designs: Let $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ be a t-(v, k, λ) design with $1 \leq t \leq k < v$. Then $\mathcal{D}$ yields further designs:

(i) The design $\mathcal{D}$ also is a $(t-1)$-$(v, k, \lambda_{\text{red}})$ design where $\lambda_{\text{red}} = \frac{v-t+1}{k-t+1}$. We call it the design with respect to smaller t, or simply the *reduced design* $\operatorname{red} \mathcal{D}$.

(ii) If x is a point of $\mathcal{V}$, the *derived design* (with respect to the point x) is $\operatorname{der}_x \mathcal{D} = (\mathcal{V} \backslash \{x\}, \mathcal{B}_x)$ where $\mathcal{B}_x = \{B \backslash \{x\} \mid B \in \mathcal{B},\ x \in B\}$. Each derived design has parameters $(t-1)$-$(v-1, k-1, \lambda)$ (regardless of the choice of the point $x \in \mathcal{V}$). Put $\lambda_{\text{der}} = \lambda$.

(iii) If x is a point of $\mathcal{V}$, the *residual design* (with respect to the point x) is $\operatorname{res}_x \mathcal{D} = (\mathcal{V} \backslash \{x\}, \mathcal{B}^x)$ where $\mathcal{B}^x = \{B \mid B \in \mathcal{B},\ x \notin B\}$. Each residual design has parameters $(t-1)$-$(v-1, k, \lambda_{\text{res}})$ where $\lambda_{\text{res}} = \lambda \frac{v-k}{k-t+1}$ (regardless of the choice of the point $x \in \mathcal{V}$). For the purpose of forming the residual design, the assumption $k < v$ is required. Note that

$$\lambda_{\text{red}} = \lambda_{\text{der}} + \lambda_{\text{res}} = \lambda + \lambda_{\text{res}}. \tag{2}$$

Since the parameter sets which we obtain are independent of the choice of x, we define the operators $\operatorname{red}, \operatorname{der}$ and res in a more abstract way, namely as maps between parameter quadruples: For t-(v, k, λ), we let $\operatorname{red} t$-(v, k, λ), $\operatorname{der} t$-(v, k, λ) and $\operatorname{res} t$-(v, k, λ) be the parameter set of the reduced, the derived and the residual design, respectively (provides these designs exist). Note that for parameters of designs, the operations derivation and forming the residual commute, since it makes no difference in which order we delete the points. Also, the reduce operator commutes with these, since considering a design as a lower t-design does not change the design itself. Hence for nonnegative integers h, i and j with $h + i + j \leq t$ and $j \leq v - k$, we can speak of the design parameter set

$$\operatorname{red}^h \operatorname{der}^i \operatorname{res}^j t\text{-}(v, k, \lambda),$$

which is obtained from t-(v, k, λ) by reducing h times, deriving i times and forming the residual j times. For the rest of this article, we are going to exploit the structure which is induced by the three operations red, der and res on the set of design parameter sets.

2 The poset of admissible design parameter sets

Not every quadruple of nonnegative integers t-(v, k, λ) is a valid parameter set of a design. Certain necessary conditions on the parameters are so fundamental, that parameter sets which satisfy these have a special name. Before we give the definition, let us introduce

$$\lambda_{\max}(t,v,k) := \binom{v-t}{k-t}, \tag{3}$$

which is the largest index λ, which a t-(v,k,λ) design may have (exactly the complete designs attain this bound).

Definition 2.1 Let t, v, k and λ be natural numbers. The parameter set t-(v,k,λ) is called *admissible* if (ADM1), (ADM2) and (ADM3-s) hold for $0 \leq s \leq t$ where

(ADM1) $t \leq k \leq v$,

(ADM2) $1 \leq \lambda \leq \lambda_{\max}(t,v,k) = \binom{v-t}{k-t}$,

(ADM3-s) $\lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} = \lambda \frac{(v-s)(v-s-1)\cdots(v-t+1)}{(k-s)(k-s-1)\cdots(k-t+1)} = \lambda_{s,0} = \lambda_{\mathrm{red}^{t-s}}$ is integral.

The last condition comes from the fact that in every t-design, and for any nonnegative integer $s \leq t$, the number $\lambda_{s,0}$ of (1) is integral. This is also the index of the $(t-s)$-fold reduced design, $\lambda_{\mathrm{red}^{t-s}}$.

A parameter set which is the parameter set of an existing design is called *realizable.* Clearly every t-design has admissible parameters but not every admissible parameter set is *realizable.* For example, Köhler in [5] shows that the admissible parameter set 13-$(32,16,3)$ is not realizable (other examples would include the parameter sets of projective planes of order 6 and 10 which are known not to exist).

For $t \leq k$, the parameters of the complete design as a t design are t-$(v,k,\lambda_{\max}(t,v,k))$. Hence for fixed t, v and k with $t \leq k \leq v$ there always is at least one admissible parameter set. The quotient

$$\lambda/\lambda_{\max}(t,v,k) \quad \left(= b/\binom{v}{k} \right),$$

which is a rational number between zero and one, describes how complete a t-(v,k,λ) design is.

Lemma 2.2 *Let D be one of the operators* red, der, res, *which is applicable to the admissible parameter set t-(v,k,λ). Let $D\big(t\text{-}(v,k,\lambda)\big) = t'$-$(v',k',\lambda')$. Then*

$$\frac{\lambda}{\lambda_{\max}(t,v,k)} = \frac{\lambda'}{\lambda_{\max}(t',v',k')}.$$

In particular, $D\big(t\text{-}(v,k,\lambda)\big)$ is complete if and only if t-(v,k,λ) is complete.

Lemma 2.3 *If t-(v,k,λ) is admissible with $t \geq 1$, then* red t-(v,k,λ) *and* der t-(v,k,λ) *are admissible, too. In addition, if $k < v$, then* res t-(v,k,λ) *is admissible as well.*

Proof: Let D be one of red, der and res, and put $D\big(t\text{-}(v,k,\lambda)\big) = t'\text{-}(v',k',\lambda')$. By Lemma 2.2, $1 \le \lambda \le \lambda_{\max}(t,v,k)$ implies $1 \le \lambda' \le \lambda_{\max}(t',v',k')$, which is (ADM2).

(i) The parameters of $\operatorname{red} t\text{-}(v,k,\lambda) = (t-1)\text{-}\Big(v,k,\lambda\frac{v-(t-1)}{k-(t-1)}\Big)$ are integral by $\big(\text{ADM3-}(t-1)\big)$. Moreover, by induction the parameters of iterated reduced designs are integral as well: $\operatorname{red}^i\operatorname{red} t\text{-}(v,k,\lambda) = \operatorname{red}^{i+1} t\text{-}(v,k,\lambda)$ is admissible for $i \le t-1$. Finally, $t-1 \le k \le v$.

(ii) $\operatorname{der} t\text{-}(v,k,\lambda) = (t-1)\text{-}(v-1,k-1,\lambda)$. For $0 \le i \le t-1$, $\operatorname{red}^i \operatorname{der} t\text{-}(v,k,\lambda) = \operatorname{der}\operatorname{red}^i t\text{-}(v,k,\lambda)$ is integral. (ADM1) is valid since $t-1 \le k-1 \le v-1$.

(iii) If $k < v$, the operator res is defined. We prove that $\operatorname{res} t\text{-}(v,k,\lambda) = (t-1)\text{-}\Big(v-1,k,\lambda\frac{v-k}{k-(t-1)}\Big)$ is admissible. Using (2) we get $\lambda_{\text{res}} = \lambda\frac{v-k}{k-(t-1)} = \lambda_{\text{red}} - \lambda$ is integral. In addition, $t-1 \le k \le v-1$. Since $(\lambda_{\text{res}})_{\text{der}} = (\lambda_{\text{der}})_{\text{res}}$ is integral by (ii), (ADM3) follows by induction.

$\square$

We deduce:

Corollary 2.4 *Let $t\text{-}(v,k,\lambda)$ be admissible. Then, for nonnegative integers h, i and j satisfying $h+i+j \le t$ and $j \le v-k$,*

$$\operatorname{red}^h\operatorname{der}^i\operatorname{res}^j t\text{-}(v,k,\lambda) = (t-i-j-h)\text{-}\Big(v-i-j,k-i,\lambda_{t-h-j,j}\Big)$$

is admissible

Hence the concept of a family makes sense:

Definition 2.5 (cf. Fig. 1) Let $t\text{-}(v,k,\lambda)$ be admissible. The *family of design parameters generated by* $t\text{-}(v,k,\lambda)$ is

$$\operatorname{Family}\big(t\text{-}(v,k,\lambda)\big) = \big\{\operatorname{red}^h \operatorname{der}^i \operatorname{res}^j t\text{-}(v,k,\lambda) \;\mid\; h,i,j \in \mathbb{N},\ h+i+j \le t,\ j \le v-k\big\}.$$

We give some more information about family members:

Theorem 2.6 *The parameter sets $t'\text{-}(v',k',\lambda')$ in the family generated by $t\text{-}(v,k,\lambda)$ are characterized by the following conditions:*

(i) $0 \le t' \le t$,

(ii) $k' \le v' \le v$,

(iii) $t' \le k' \le k$,

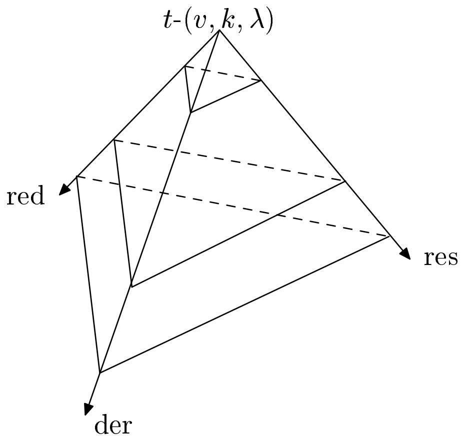


Figure 1: The family of a t-(v,k,λ)

(iv) $k-k' \le v-v' \le t-t'$,

(v) $\lambda' = \lambda_{k-k'+t',v-v'-k+k'}$.

Proof: Let t'-(v',k',λ') be a parameter set satisfying (i)-(v). Then $i := k-k'$ and $j := v-v'-(k-k')$ and $h := t-t'-i-j = t-t'-(v-v')$ are nonnegative integers with $h+i+j = t-t' \le t$, and $j \le v-k-(v'-k') \le v-k$. By Lemma 2.4,

$$\begin{aligned}\mathrm{red}^h \mathrm{der}^i \mathrm{res}^j\, t\text{-}(v,k,\lambda) &= (t-i-j-h)\text{-}\big(v-i-j, k-i, \lambda_{t-h-j,j}\big)\\ &= t'\text{-}(v',k',\lambda_{t'+(k-k'),v-v'-(k-k')})\end{aligned}$$

and thus t'-$(v',k',\lambda') \in \mathrm{Family}\big(t\text{-}(v,k,\lambda)\big)$. A routine check using Lemma 2.4 shows that all design parameter sets contained in the family of a t-(v,k,λ) satisfy (i)-(v). □

Consider the following question: given t, k and v, what are the possible values of λ in admissible t-(v,k,λ)? Before we can answer this, let us introduce the number

$$\Delta\lambda(t,v,k) := \mathrm{lcm}\left\{ \frac{\binom{k-s}{t-s}}{\gcd\left(\binom{k-s}{t-s},\binom{v-s}{t-s}\right)} \;\middle|\; 0 \le s \le t \right\}. \tag{4}$$

Then

$$\begin{aligned}\Delta\lambda(t,v,k) =& \Delta\lambda(t,v,v-k),\\ \Delta\lambda(k,v,k) =& \Delta\lambda(0,v,k) = \Delta\lambda(t,v,v) = 1\end{aligned}$$

for all $t \leq k \leq v$. The following result gives a characterization of admissible parameter sets. In particular, it shows that given t, k and v, the smallest index λ for an admissible parameter set t-(v,k,λ) is $\Delta\lambda(t,v,k)$.

Proposition 2.7 *Let t, v, k and λ be nonnegative rationals with $k < v$. The following conditions are equivalent:*

(i) *The parameter set t-(v,k,λ) is admissible.*

(ii) $\mathrm{red}^i\, t$-(v,k,λ) *is integral for* $0 \leq i \leq t$, *where* $t \leq k$ *and* $1 \leq \lambda \leq \lambda_{\max}(t,v,k)$.

(iii) $t, v, k, \lambda \in \mathbb{N}$, $\Delta\lambda(t,v,k) \mid \lambda$, $t \leq k$ *and* $1 \leq \lambda \leq \lambda_{\max}(t,v,k)$.

(iv) $\mathrm{der}\, t$-(v,k,λ) *and* $\mathrm{res}\, t$-(v,k,λ) *are admissible.*

(v) $\mathrm{red}\, t$-(v,k,λ) *and* $\mathrm{der}\, t$-(v,k,λ) *are admissible.*

(vi) $\mathrm{red}\, t$-(v,k,λ) *and* $\mathrm{res}\, t$-(v,k,λ) *are admissible and* $t' < k'$ *where* $\mathrm{res}\, t$-$(v,k,\lambda) = t'$-(v',k',λ').

Proof:

$(i) \Leftrightarrow (ii)$: As $\mathrm{res}^i\, t$-$(v,k,\lambda) = s$-$\left(v, k, \lambda\frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}\right)$ with $s := t - i$, (i) and (ii) are equivalent.

$(i) \Leftrightarrow (iii)$: If t-(v,k,λ) is admissible, then for $0 \leq s \leq t$, the number $\lambda\frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$ is integral. Thus $\binom{k-s}{t-s}$ divides $\lambda\binom{v-s}{t-s}$ which implies that $\frac{\binom{k-s}{t-s}}{\gcd\left(\binom{k-s}{t-s},\binom{v-s}{t-s}\right)}$ divides λ for all these s. Therefore $\Delta\lambda$ divides λ. The other implication is clear.

$(i) \Leftrightarrow (iv)$: By Lemma 2.3, (i) implies (iv). On the other hand, assume that $\mathrm{der}\, t$-(v,k,λ) and $\mathrm{res}\, t$-(v,k,λ) are admissible. In particular, $\lambda_{\mathrm{res}} = \lambda\frac{v-k}{k-(t-1)}$ is integral. Then (2) implies $\lambda_{\mathrm{red}} = \lambda_{\mathrm{der}} + \lambda_{\mathrm{res}}$ is integral, so $\mathrm{red}\, t$-(v,k,λ) is integral. It remains to show that λ_{red^i} is integral for $1 < i \leq t$. If $t = 1$, there is nothing to show. So assume $t \geq 2$. We apply (2) to get $\lambda_{\mathrm{red}^2} = (\lambda_{\mathrm{red}})_{\mathrm{der}} + (\lambda_{\mathrm{red}})_{\mathrm{res}} = (\lambda_{\mathrm{der}})_{\mathrm{red}} + (\lambda_{\mathrm{res}})_{\mathrm{red}}$, parameters which are integral by our assumption. So λ_{red^2} is integral. We can proceed by induction.

$(i) \Leftrightarrow (v)$: By Lemma 2.3, (i) implies (v). Now assume that $\mathrm{red}\, t$-(v,k,λ) and $\mathrm{der}\, t$-(v,k,λ) are admissible. In particular, $\lambda_{\mathrm{der}} = \lambda$ is integral. Moreover, $t \leq k$ since $t - 1 \leq k - 1$ holds for the derived parameter set. Also, $\lambda \leq \lambda_{\max}(v,t,k)$ by Lemma 2.2. The conditions (ADM3-s) for $0 \leq s \leq t-1$ are satisfied, and hence t-(v,k,λ) is admissible.

$(i) \Leftrightarrow (vi)$: Again by Lemma 2.3, (i) implies (vi). Now assume that $\mathrm{red}\, t$-(v,k,λ) and $\mathrm{res}\, t$-(v,k,λ) are admissible. By (2), $\lambda = \lambda_{\mathrm{der}} = \lambda_{\mathrm{red}} - \lambda_{\mathrm{res}}$ is integral which is (ADM3-t). The assumption $t' < k'$

implies $t-1 < k$, hence $t \leq k < v$. (ADM2) follows by Lemma 2.2. The conditions (ADM3-s) for $0 \leq s \leq t-1$ correspond to the conditions for the parameter set $\mathrm{red}\, t\text{-}(v,k,\lambda)$ for $s > 0$.

□

Since the parameters of the complete design are admissible, the previous result implies that

$$\Delta\lambda(t,v,k) \;\Big|\; \lambda_{\max}(t,v,k). \tag{5}$$

It is useful to introduce the *poset of admissible design parameters*, denoted as $\mathcal{P}$, as the transitive closure of the relationships induced by the operators der, red and res on admissible parameter sets. This means that we have

$$t'\text{-}(v',k',\lambda') \;\leq\; t\text{-}(v,k,\lambda)$$

if and only if there is a sequence $D_1,\ldots,D_r$ of operators chosen from der, red and res such that

$$D_r(\cdots(D_1(t\text{-}(v,k,\lambda)))\cdots) = t'\text{-}(v',k',\lambda').$$

Note that the families of Definition 2.5 are just the order theoretic ideal in the poset $\mathcal{P}$.

The question arises whether there exists a largest family containing a given parameter set. In terms of the poset $\mathcal{P}$, this questions amounts to whether or not there always is a maximal element above any given element. The purpose of this section is to settle this question. We note that the complete design is the derived design of an infinite number of complete designs with larger block size. This means we will have to exclude complete designs from our consideration.

We introduce the inverse operators red^{-1}, der^{-1} and res^{-1} (cf. Fig. 2). These are only partially

$$\begin{tikzcd}
(t+1)\text{-}(v,k,\lambda\frac{k-t}{v-t}) & (t+1)\text{-}(v+1,k+1,\lambda) & (t+1)\text{-}(v+1,k,\lambda\frac{k-t}{v+1-k}) \\
& t\text{-}(v,k,\lambda) \arrow[ul, "\mathrm{red}^{-1}"] \arrow[u, "\mathrm{der}^{-1}"] \arrow[ur, "\mathrm{res}^{-1}"] \arrow[dl, "\mathrm{red}"] \arrow[d, "\mathrm{der}"] \arrow[dr, "\mathrm{res}"] & \\
(t-1)\text{-}(v,k,\lambda\frac{v-(t-1)}{k-(t-1)}) & (t-1)\text{-}(v-1,k-1,\lambda) & (t-1)\text{-}(v-1,k,\lambda\frac{v-k}{k-(t-1)})
\end{tikzcd}$$

Figure 2: The operators $\mathrm{red}^{\pm 1}$, $\mathrm{der}^{\pm 1}$ and $\mathrm{res}^{\pm 1}$

defined functions as we require the image to be admissible:

Definition 2.8 Let t-(v,k,λ) be an admissible parameter set. Put

(i) $\mathrm{red}^{-1} t$-$(v,k,\lambda) := (t+1)$-$(v,k,\lambda\frac{k-t}{v-t})$ if admissible

(ii) $\mathrm{der}^{-1} t$-$(v,k,\lambda) := (t+1)$-$(v+1,k+1,\lambda)$ if admissible and

(iii) $\mathrm{res}^{-1} t$-$(v,k,\lambda) := (t+1)$-$(v+1,k,\lambda\frac{k-t}{v+1-k})$ if admissible.

If one of these functions is defined, we say that the given parameter set *extends* under that operator.

Let us return to the study of maximal elements in the poset $\mathcal{P}$. A related — but much harder — problem is to determine whether a design can be extended, i.e. whether there exists another design whose derived design is the given one (for example, Cameron in [3] determines which square designs are extendible). Of course, for a design to be extended, the parameter set of the extension must be admissible, i.e. the operator red^{-1} must be defined. Hence admissibility of parameter sets give necessary conditions for extensions of designs. Let us quote two results in this context, which were mentioned by Dembowski [4, p. 76, 77]. We should recall that the parameters $\lambda_{i,j}$ of (1) are integral for admissible parameter sets.

Lemma 2.9 *Let t-(v,k,λ) be an admissible t-design parameter set. Recall that $b=\lambda_{0,0}$ denotes the number of blocks of a design, and $r=\lambda_{1,0}$ is the number of blocks on a point.*

(i) A necessary condition for der^{-1} to be defined is that $b(v+1)$ is divisible by $k+1$.

(ii) Assume $t\geq 2$. A necessary condition for res^{-1} to be defined is that $\lambda_{2,0}(k-2)$ is divisible by $v+1-k$.

Note that Alltop [1] describes further conditions under which t-designs can be extended.

The following analogue of Lemma 2.2 is easily proved:

Lemma 2.10 *Let t-(v,k,λ) be an admissible parameter set. Assume that $D\big(t$-$(v,k,\lambda)\big) = t'$-(v',k',λ') is admissible for some $D\in\{\mathrm{red}^{-1},\mathrm{der}^{-1},\mathrm{res}^{-1}\}$. Then*

$$\frac{\lambda}{\lambda_{\max}(t,v,k)} = \frac{\lambda'}{\lambda_{\max}(t',v',k')}.$$

From this we deduce that

$$\frac{\lambda'}{\lambda} = \frac{\lambda_{\max}(t',v',k')}{\lambda_{\max}(t,v,k)} = f^D_{t,v,k}, \quad \text{where} \quad f^D_{t,v,k} = \begin{cases} \frac{k-t}{v-t} & \text{if } D=\mathrm{red}^{-1}, \\ 1 & \text{if } D=\mathrm{der}^{-1}, \\ \frac{k-t}{v+1-k} & \text{if } D=\mathrm{res}^{-1}. \end{cases} \tag{6}$$

Note that $f^D_{t,v,k}$ is just the factor by which the index changes under the operator D.

The next result follows from commutativity of the six operators $\{\mathrm{red}^{\pm1},\mathrm{der}^{\pm1},\mathrm{res}^{\pm1}\}$:

Lemma 2.11 *Let t-(v,k,λ) be an admissible design parameter set, and assume that $\mathrm{red}^{-h}\mathrm{der}^{-i}\mathrm{res}^{-j}\big(t\text{-}(v,k,\lambda)\big)$ is admissible for some nonnegative integers h, i and j. Then $\mathrm{red}^{-h'}\mathrm{der}^{-i'}\mathrm{res}^{-j'}\big(t\text{-}(v,k,\lambda)\big)$ is admissible for all nonnegative integers $h' \le h$, $i' \le i$ and $j' \le j$.*

Lemma 2.12 *Let t-(v,k,λ) be admissible, and assume that $D_1\big(t\text{-}(v,k,\lambda)\big)$ and $D_2\big(t\text{-}(v,k,\lambda)\big)$ are defined for $D_1, D_2 \in \{\mathrm{red}^{-1}, \mathrm{der}^{-1}, \mathrm{res}^{-1}\}$, $D_1 \neq D_2$. In addition, if $\{D_1, D_2\} = \{\mathrm{red}^{-1}, \mathrm{res}^{-1}\}$ we assume that $\lambda \neq \lambda_{\max}$. Then $D_1D_2\big(t\text{-}(v,k,\lambda)\big) = D_2D_1\big(t\text{-}(v,k,\lambda)\big)$ is admissible, too.*

Proof: We distinguish 3 cases according to D_1, D_2. Up to a reordering of D_1 and D_2 these are all possible cases.

$D_1 = \mathrm{der}^{-1}, D_2 = \mathrm{res}^{-1}$: We do not yet know if $\mathrm{der}^{-1}\mathrm{res}^{-1}\big(t\text{-}(v,k,\lambda)\big)$ is admissible, but the commutativity of the operators allows to deduce

$$\mathrm{res}\big(\mathrm{der}^{-1}\mathrm{res}^{-1}t\text{-}(v,k,\lambda)\big) = \mathrm{der}^{-1}t\text{-}(v,k,\lambda)$$

and

$$\mathrm{der}\big(\mathrm{der}^{-1}\mathrm{res}^{-1}t\text{-}(v,k,\lambda)\big) = \mathrm{res}^{-1}t\text{-}(v,k,\lambda)$$

are admissible by assumption. Hence by Proposition 2.7, (iv) $\Leftrightarrow$ (i), $\mathrm{der}^{-1}\mathrm{res}^{-1}t\text{-}(v,k,\lambda)$ is admissible, i. e. $D_1D_2\big(t\text{-}(v,k,\lambda)\big)$ is defined.

$D_1 = \mathrm{red}^{-1}, D_2 = \mathrm{der}^{-1}$: We can proceed in a similar way using Proposition 2.7, (v) $\Leftrightarrow$ (i), respectively, to get the result in that case.

$D_1 = \mathrm{red}^{-1}, D_2 = \mathrm{res}^{-1}$: In this case we have the additional assumption $\lambda \neq \lambda_{\max}$ which we need to show that $\mathrm{red}^{-1}\mathrm{res}^{-1}\big(t\text{-}(v,k,\lambda)\big)$ satisfies (ADM1): Deny this. Then $t+1 = k$ and $\mathrm{red}^{-1}t\text{-}(v,t+1,\lambda) = (t+1)\text{-}\big(v,t+1,\lambda\frac{1}{v-t}\big)$ and $\mathrm{res}^{-1}t\text{-}(v,t+1,\lambda) = (t+1)\text{-}\big(v+1,t+1,\lambda\frac{1}{v-t}\big)$. But the last two parameter sets are complete, hence t-(v,k,λ) is complete by Lemma 2.10, contradicting the assumption $\lambda \neq \lambda_{\max}$. We conclude that $t \le k-2$, so $\mathrm{red}^{-1}\mathrm{res}^{-1}\big(t\text{-}(v,k,\lambda)\big)$ satisfies (ADM1). We proceed as usual:

$$\mathrm{res}\big(\mathrm{red}^{-1}\mathrm{res}^{-1}t\text{-}(v,k,\lambda)\big) = \mathrm{red}^{-1}t\text{-}(v,k,\lambda)$$

and

$$\mathrm{red}\big(\mathrm{red}^{-1}\mathrm{res}^{-1}t\text{-}(v,k,\lambda)\big) = \mathrm{res}^{-1}t\text{-}(v,k,\lambda)$$

are admissible by assumption. Moreover, $t+1 < k$, which means that the two parameter sets on the right hand side satisfy the additional assumption of Proposition 2.7, (vi). Hence $\mathrm{red}^{-1}\mathrm{res}^{-1}t\text{-}(v,k,\lambda)$ is admissible, i. e. $D_1D_2\big(t\text{-}(v,k,\lambda)\big)$ is defined. □

Lemma 2.13 *Let t-(v,k,λ) be an admissible design parameter set with $\lambda \neq \binom{v-t}{k-t}$. Assume that for nonnegative integers h_1, h_2, i_1, i_2, j_1, j_2 the parameter sets*

$$\mathrm{red}^{-h_\ell}\,\mathrm{der}^{-i_\ell}\,\mathrm{res}^{-j_\ell}\,t\text{-}(v,k,\lambda) \ \ \textit{are admissible for} \ \ \ell = 1,2. \tag{7}$$

Then $\mathrm{red}^{-\max(h_1,h_2)}\,\mathrm{der}^{-\max(i_1,i_2)}\,\mathrm{res}^{-\max(j_1,j_2)}\,t$-$(v,k,\lambda)$ *is admissible as well.*

Proof: First note that $\mathrm{red}^{-\min(h_1,h_2)}\,\mathrm{der}^{-\min(i_1,i_2)}\,\mathrm{res}^{-\min(j_1,j_2)}\,t$-$(v,k,\lambda)$ is admissible by Lemma 2.11. Hence it suffices to prove the claim for the case that one of the corresponding integers is zero, i. e. $h_1h_2 = 0$, $i_1i_2 = 0$ and $j_1j_2 = 0$. The assumption is still given by (7). Assume that $h_2 \neq 0$, hence $h_1 = 0$. In this case we pick red^{-1} and use Lemma 2.12 to show that $\mathrm{red}^{-1}\mathrm{der}^{-i_1}\mathrm{res}^{-j_1}\,t$-$(v,k,\lambda)$ is admissible. By Lemma 2.11, $\mathrm{der}^{-i_1}\mathrm{res}^{-j_1}\big(\mathrm{red}^{-1}t\text{-}(v,k,\lambda)\big)$ is admissible, as well as $\mathrm{der}^{-i_2}\mathrm{res}^{-j_2}\big(\mathrm{red}^{-1}t\text{-}(v,k,\lambda)\big)$, i. e. the assumption (7) holds for $\mathrm{red}^{-1}t$-(v,k,λ) instead of t-(v,k,λ) and h_2 reduced by one. We proceed by induction. Similarly, we proceed with the other operators. The assumption $\lambda \neq \binom{v-t}{k-t}$ is needed for applying Lemma 2.12. □

Theorem 2.14 *Let t-(v,k,λ) be an admissible parameter set with $\lambda \neq \binom{v-t}{k-t}$. Then there exists a unique largest admissible parameter set, called* $\mathrm{Ancestor}\big(t\text{-}(v,k,\lambda)\big)$, *such that t-(v,k,λ) is contained in its family. More precisely, there exist nonnegative integers $h_{\max}$, $i_{\max}$ and $j_{\max}$ maximal with respect to the property that*

$$\mathrm{Ancestor}\big(t\text{-}(v,k,\lambda)\big) = \mathrm{red}^{-h_{\max}}\,\mathrm{der}^{-i_{\max}}\,\mathrm{res}^{-j_{\max}}\,t\text{-}(v,k,\lambda)$$

is defined. The given parameter set is the $h_{\max}$-fold reduction, $i_{\max}$-fold derived and $j_{\max}$-fold residual of its ancestor. A design parameter set which equals its own ancestor is called ancestor parameter set. *If* $\mathrm{Ancestor}\big(t\text{-}(v,k,\lambda)\big) = t'$-$(v',k',\lambda')$ *then*

$$\frac{\lambda'}{\lambda_{\max}(t',v',k')} = \frac{\lambda}{\lambda_{\max}(t,v,k)}.$$

Hence the ancestor is again incomplete.

Proof: Fix an admissible parameter set t-(v,k,λ) with $\lambda < \binom{v-t}{k-t} = \lambda_{\max}$. We first have to show that $\mathrm{Ancestor}\big(t\text{-}(v,k,\lambda)\big)$ is defined, i. e. that the integers $h_{\max}$, $i_{\max}$ and $j_{\max}$ exist. Therefore, we look at the parameters arising as results of the operations der^{-1}, red^{-1} and res^{-1} (cf. Fig. 2). In case of red^{-1} and res^{-1}, the difference $k-t$ strictly decreases, so $i_{\max}$ and $j_{\max}$ are both bounded above by $k-t$. What can be said about the number of times that der^{-1} can be applied? Assume this is the case infinitely often and put

$$b_n = \#\text{ of blocks of } \mathrm{der}^{-n}\,t\text{-}(v,k,\lambda) = \lambda\frac{\binom{v+n}{t+n}}{\binom{k+n}{t+n}}.$$

By (ADM3-0), b_n is an integer. Note that

$$\frac{b_{n+1}}{b_n} = \frac{v+n+1}{k+n+1} \tag{8}$$

holds, even for the complete design with $\lambda = \lambda_{\max}(t,v,k)$. For simplicity, we write $\lambda_{\max}$ instead of $\lambda_{\max}(t,v,k)$. Put

$$c_n = \#\text{ of blocks of } \mathrm{der}^{-n}\, t\text{-}(v,k,\lambda_{\max}) = \lambda_{\max}\frac{\binom{v+n}{t+n}}{\binom{k+n}{t+n}} = \binom{v+n}{k+n}.$$

By (8), the sequences of numbers (b_n) and (c_n) are proportional. Hence there exists a rational number γ with $0<\gamma<1$ and $\gamma = \frac{b_n}{c_n}$ for all $n \geq 0$. In particular,

$$\gamma = \frac{b_0}{c_0} = \frac{\binom{v}{t}\lambda}{\binom{k}{t}\binom{v}{k}} = \frac{\binom{v}{t}\lambda}{\binom{v}{t}\binom{v-t}{k-t}} = \frac{\lambda}{\lambda_{\max}}.$$

Moreover,

$$b_n = \gamma c_n = \frac{\lambda\binom{v+n}{k+n}}{\lambda_{\max}} \in \mathbb{N}$$

for all integers $n \geq 0$. Let p be a prime dividing $\frac{\lambda_{\max}}{\gcd(\lambda_{\max},\lambda)}$ (such a prime exists since $0<\lambda<\lambda_{\max}$ by assumption). We deduce that p divides $\binom{v+n}{k+n}$ for all n. But this is impossible as for example $\binom{p^m-1}{j} = \prod_{h=1}^{j}\frac{p^m-h}{h} \not\equiv 0 \mod p$ for all $0 \leq j \leq p^m-1$ and arbitrary m : If the numerator is divisible by p^s, say, then the denominator is divisible by that number, too. Hence all factors p cancel in the product. We conclude that the number of times that red^{-1} can be applied is finite, and we let $h_{\max}$ be maximal with respect to the property that $\mathrm{res}^{-h_{\max}}$ is defined.

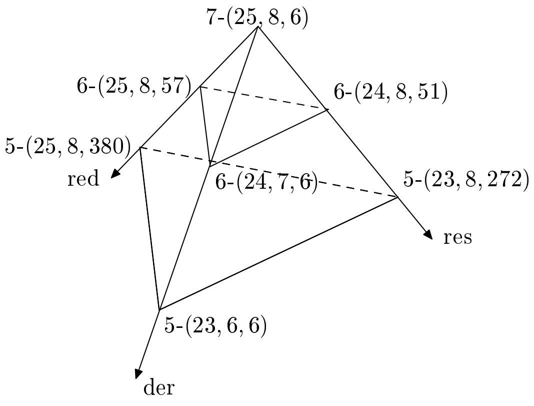
Lemma 2.13 implies the uniqueness of the ancestor parameter set. The final two statements follow by repeated application of Lemma 2.10. □

Example 2.15 There exist 5-$(23,6,6)$ designs invariant under $\mathrm{Hol}(\mathrm{C}_{23})$. We find that

$$\mathrm{der}^{-2}\, 5\text{-}(23,6,6) = 7\text{-}(25,8,6)$$

is ancestor. (Note that it is not known whether or not a 7-$(25,8,6)$ design exists.) Figure 3 displays the family of this parameter set. On the left, the pyramid of parameter sets with $t \geq 5$ is shown Note that in the bottom layer, there are three more design parameter sets. A more concise way of displaying the family is indicated to the right, which shows the layers one after another. The underlined parameter sets are known to be realizable. ◇

As an application, we evaluate the ancestor for Steiner systems $S(t,t+1,v)$ with $v-t$ prime:



$$\frac{\text{7-}(25,8,6)}{\begin{array}{ll}\text{6-}(25,8,57) & \text{6-}(24,8,51)\\ \text{6-}(24,7,6) & \end{array}}$$

<u>5-(25,8,380)</u>	<u>5-(24,8,323)</u>	5-(23, 8, 272)
<u>5-(24,7,57)</u>	5-(23, 7, 51)	
<u>5-(23,6,6)</u>		

Figure 3: The family of 7-(25, 8, 6)

Proposition 2.16 *Let t and v be integers with $v - t$ a prime. Then*

$$\text{Ancestor}(t\text{-}(v, t+1, 1)) = \text{der}^{-n}\, t\text{-}(v, t+1, 1) = (t+n) - \big(v+n, t+n+1, 1\big)$$

with $n = v - 2(t+1)$. In particular, if $\text{Ancestor}(t\text{-}(v, t+1, 1)) = t'\text{-}(v', k', 1)$ *then* $\frac{k'}{v'} = \frac{1}{2}$.

Proof:

(i) We first show that $(t+n)\text{-}\big(v+n, t+n+1, 1\big)$ is admissible if and only if $n \leq v - 2(t+1)$. We have the following equivalence:

$$\begin{aligned} & (t+n)\text{-}\big(v+n, t+n+1, 1\big) \text{ is admissible} \\ \Longleftrightarrow\ & (\text{ADM3-}s) \text{ holds for } 0 \leq s \leq t+n \\ \Longleftrightarrow\ & (\text{ADM3-}(t+n-s)) \text{ holds for } 0 \leq s \leq t+n \\ \Longleftrightarrow\ & \frac{(v-(t-1))\cdots(v-(t-s))}{2\cdots(s+1)} = \frac{1}{v-t}\binom{v+t+s}{s+1} \text{ is integral for } 0 \leq s \leq t+n. \end{aligned}$$

If $s+1 < v-t$ then $v-t$ prime implies $\gcd\big((s+1)!, v-t\big) = 1$. Since $\binom{v+t+s}{s+1}$ is an integer for all s,

$$(s+1)! \;\Big|\; (v-(t-1))\cdots(v-(t-s))$$

for $s+1 < v-t \Leftrightarrow s \leq v-t-2$. Hence $(t+n)\text{-}\big(v+n, t+n+1, 1\big)$ is admissible for all n satisfying $t+n \leq v-t-2 \Leftrightarrow n \leq v-2(t-1)$. On the other hand, if $n = v-2t-1$ then the parameter set in question would be $(v-t-1)$-$(2v-2t-1, v-t, 1)$, so $\bigl(\text{ADM3-0}\bigr)$ would require that

$$\frac{(2v-2t-1-(v-t-2))\cdots(2v-2t-1)}{(v-t-(v-t-2))\cdots(v-t-0)} = \frac{(v-t+1)\cdots(2(v-t)-1)}{2\cdots(v-t)}$$

were integral. But the prime $v-t$ divides none of the factors in the numerator, a contradiction.

(ii) Part (i) implies that $\mathrm{der}^{-n}\, t-(v, t+1, 1) = (t+n)\text{-}\bigl(v+n, t+n+1, 1\bigr)$ is defined for $n = v-2(t+1)$ and no larger n. We claim that it is not possible to apply red^{-1} or res^{-1} to this parameter set: If red^{-1} were applicable, then $(v+n)-(t+n-1) = v-t+1$ had to divide $1\cdot(t+n+1-(t+n)) = 1$ which it does not (since $v-t \neq 0$). If res^{-1} were applicable, then $v+n+1-(t+n+1) = v-t$ had to divide 1 which it does not (since $v-t\neq 1$). Hence, $\mathrm{Ancestor}\bigl(t-(v, t+1, 1)\bigr) = \mathrm{der}^{-n}\, t-(v, t+1, 1)$ with $n = v-2(t+1)$. In particular, if $\mathrm{Ancestor}\bigl(t-(v, t+1, 1)\bigr) = t'-(v', k', 1) = (t+n)\text{-}\bigl(v+n, t+n+1, 1\bigr)$, then

$$\frac{k'}{v'} = \frac{t+n+1}{v+n} = \frac{v-t-1}{2v-2t-2} = \frac{1}{2}.$$

□

For example, $\mathrm{Ancestor}\bigl(5\text{-}(244, 6, 1)\bigr) = \mathrm{der}^{-232}\bigl(5\text{-}(244, 6, 1)\bigr) = 237\text{-}(476, 238, 1)$.

3 Which design parameter sets extend?

The ancestors of parameter sets differing only in the index can look quite different:

Example 3.1 Consider the collection of parameter sets of the form $5\text{-}(24, 8, \lambda)$ where $1 \le \lambda \le \lambda_{\max}(5, 24, 8) = 969 = 3\cdot 17\cdot 19$ (note that $\Delta\lambda(5, 24, 8) = 1$). We get

$$\mathrm{Ancestor}\bigl(5\text{-}(24,8,\lambda)\bigr) = \begin{cases} 17\text{-}(36,18,m\cdot 1) = \mathrm{red}^{-0}\mathrm{der}^{-10}\mathrm{res}^{-2}\, 5\text{-}(24,8,m\cdot 51) & \text{if } \lambda = m\cdot 51 \\ 15\text{-}(32,16,m\cdot 1) = \mathrm{red}^{-2}\mathrm{der}^{-8}\mathrm{res}^{-0}\, 5\text{-}(24,8,m\cdot 57) & \text{if } \lambda = m\cdot 57 \\ 13\text{-}(32,16,m\cdot 3) = \mathrm{red}^{-0}\mathrm{der}^{-8}\mathrm{res}^{-0}\, 5\text{-}(24,8,m\cdot 3) & \text{if } \lambda = m\cdot 3,\ 51\nmid\lambda,\ 57\nmid\lambda \\ 6\text{-}(25,8,m\cdot 3) = \mathrm{red}^{-0}\mathrm{der}^{-0}\mathrm{res}^{-1}\, 5\text{-}(24,8,m\cdot 17) & \text{if } \lambda = m\cdot 17, 19\nmid\lambda \\ 6\text{-}(24,8,m\cdot 3) = \mathrm{red}^{-1}\mathrm{der}^{-0}\mathrm{res}^{-0}\, 5\text{-}(24,8,m\cdot 19) & \text{if } \lambda = m\cdot 19, 17\nmid\lambda \\ 7\text{-}(25,8,m\cdot 6) = \mathrm{red}^{-1}\mathrm{der}^{-0}\mathrm{res}^{-1}\, 5\text{-}(24,8,m\cdot 323) & \text{if } \lambda = m\cdot 323 = 17\cdot 19 \\ 5\text{-}(24,8,\lambda) = \mathrm{red}^{-0}\mathrm{der}^{-0}\mathrm{res}^{-0}\, 5\text{-}(24,8,\lambda) & \text{otherwise} \end{cases}$$

Note that the "otherwise" case is equivalent to

$$\gcd\left(\lambda, \frac{\lambda_{\max}(5,24,8)}{\Delta\lambda(5,24,8)}\right) = \gcd(\lambda, 969/1) = \gcd(\lambda, 3\cdot 17\cdot 19) = 1.$$

◇

In order to determine the ancestor, we have to find out which parameter sets extend under the operators red^{-1}, der^{-1} or res^{-1}. For this, we examine properties of the function $\Delta\lambda$.

Proposition 3.2 *Let v and k be integers with $k \le v$. Then*

(i)

$$\Delta\lambda(t+1, v, k) = \frac{k-t}{\gcd(v-t, k-t)} \cdot \frac{\Delta\lambda(t, v, k)}{\gcd\left(\Delta\lambda(t, v, k), \frac{v-t}{\gcd(v-t,k-t)}\right)} \quad \textit{for all } t < k, \tag{9}$$

(ii)

$$\Delta\lambda(t+1, v+1, k+1) = \mathrm{lcm}\left(\Delta\lambda(t, v, k), \frac{\binom{k+1}{t+1}}{\gcd\left(\binom{k+1}{t+1}, \binom{v+1}{t+1}\right)}\right). \tag{10}$$

Note that $\frac{\binom{k+1}{t+1}}{\gcd\left(\binom{k+1}{t+1}, \binom{v+1}{t+1}\right)}$ *is the smallest natural number a such that* $a\frac{(v+1)\cdots(v-t+1)}{(k+1)\cdots(k-t+1)}$ *is integral.*

Proof:

(i) By definition, $\Delta\lambda(t, v, k)$ is the smallest natural number such that (ADM3-0) $\wedge \cdots \wedge$ (ADM3-t) hold for the parameter set t-$\big(v, k, \Delta\lambda(t, v, k)\big)$. Consider the corresponding admissibility condition for the parameters $(t+1)$-$\big(v, k, \Delta\lambda(t+1, v, k)\big)$, which we denote by (ADM3'-0) $\wedge \cdots \wedge$ (ADM3'-$(t+1)$). (ADM3'-$(t+1)$) just implies that $\Delta\lambda(t+1, v, k)$ is an integer. Write

$$\frac{v-t}{k-t} = \frac{\frac{v-t}{\gcd(v-t,k-t)}}{\frac{k-t}{\gcd(v-t,k-t)}}$$

with coprime numerator and denominator. Since (ADM3'-t) states that $\Delta\lambda(t+1, v, k)\frac{v-t}{k-t}$ is integral, $\Delta\lambda(t+1, v, k)$ must be a multiple of $\frac{k-t}{\gcd(k-t,v-t)}$. Put

$$L^{(t+1)} := \frac{\Delta\lambda(t+1, v, k)}{\frac{k-t}{\gcd(v-t,k-t)}} \in \mathbb{N}$$

and consider the two sets of conditions in parallel. For $0 \le s \le t-1$, we have

$$\begin{aligned}
\text{(ADM3-}s\text{)} &\iff \Delta\lambda(t, v, k)\frac{(v-s)\cdots(v-(t-1))}{(k-s)\cdots(k-(t-1))} \in \mathbb{N}, \\
\text{(ADM3'-}s\text{)} &\iff \Delta\lambda(t+1, v, k)\frac{(v-s)\cdots(v-(t-1))(v-t)}{(k-s)\cdots(k-(t-1))(k-t)} \in \mathbb{N} \\
&\iff L^{(t+1)} \cdot \frac{(v-s)\cdots(v-(t-1))\frac{v-t}{\gcd(v-t,k-t)}}{(k-s)\cdots(k-(t-1))} \in \mathbb{N}.
\end{aligned}$$

Note that the additional factor $\frac{v-t}{k-t}$ in (ADM3'-s) does not depend on s. The last integrality condition shows that

$$\frac{\Delta\lambda(t,v,k)}{\gcd\left(\Delta\lambda(t,v,k),\frac{v-t}{\gcd(v-t,k-t)}\right)}$$

is the smallest solution for $L^{(t+1)}$. Hence by definition of $L^{(t+1)}$,

$$\Delta\lambda(t+1,v,k)=\frac{k-t}{\gcd(v-t,k-t)}\cdot\frac{\Delta\lambda(t,v,k)}{\gcd\left(\Delta\lambda(t,v,k),\frac{v-t}{\gcd(v-t,k-t)}\right)}.$$

(ii) For $0\le s\le t$, let (ADM3-s) be the admissibility condition for the t-$(v,k,\Delta\lambda)$ design and for $0\le s\le t+1$, let (ADM3'-s) be the admissibility condition for the $(t+1)$-$(v+1,k+1,\lambda)$ design. Note that (ADM3-s) $\iff$ (ADM3'-$(s+1)$) for $0\le s\le t$ while (ADM3'-0) requires that

$$\Delta\lambda(t+1,v+1,k+1)\frac{\binom{v+1}{t+1}}{\binom{k+1}{t+1}}$$

is integral. This last condition can be reworded as

$$\frac{\binom{k+1}{t+1}}{\gcd\left(\binom{k+1}{t+1},\binom{v+1}{t+1}\right)}\;\Bigg|\;\Delta\lambda(t+1,v+1,k+1).$$

Hence $\Delta\lambda(t+1,v+1,k+1)$ is the least common multiple of $\Delta\lambda(t,v,k)$ and that number.

□

Example 3.3 (cf. Example 3.1) Consider the parameter set 5-$(24,8,1)$ again. We have:

(i) $\Delta\lambda(6,24,8)=\frac{3}{\gcd(19,3)}\cdot\frac{1}{\gcd\left(1,\frac{19}{\gcd(19,3)}\right)}=3$;

(ii) $\Delta\lambda(6,25,9)=\mathrm{lcm}(1,3)=3$ as $\frac{25}{9}\frac{24}{8}\frac{23}{7}\frac{22}{6}\frac{21}{5}\frac{20}{4}=\frac{25\cdot23\cdot11}{3}$, hence 3 is the smallest natural number making this fraction integral;

(iii) $\Delta\lambda(6,25,8)=3$ since $a=3$ is the smallest natural number such that all prefixing partial products in $a\cdot\frac{20}{3}\cdot\frac{21}{4}\cdot\frac{22}{5}\cdot\frac{23}{6}\cdot\frac{24}{7}\cdot\frac{25}{8}$ evaluate to integers.

◇

The question of whether an admissible design parameter set t-(v,k,λ) extends under one of the operators red^{-1}, der^{-1} or res^{-1} turns out to be equivalent to certain divisibility conditions for λ in terms of t, v and k. Our results are strongest in the case of the operators red^{-1} and der^{-1} since we have the recursion formulae (9) and (10) of Proposition 3.2 in that cases.

Proposition 3.4 *Let t-(v,k,λ) be an admissible design parameter set (hence $\frac{\lambda}{\Delta\lambda(t,v,k)}$ is an integer by (4)). Then*

(i) $\mathrm{red}^{-1}\, t$-(v,k,λ) *is defined if and only if*

(a) $t < k$ *and*

(b) $c^{\mathrm{red}^{-1}}_{t,v,k} \,\big|\, \frac{\lambda}{\Delta\lambda(t,v,k)}$ *where*

$$c^{\mathrm{red}^{-1}}_{t,v,k} := \frac{\frac{v-t}{\gcd(v-t,k-t)}}{\gcd\left(\Delta\lambda(t,v,k), \frac{v-t}{\gcd(v-t,k-t)}\right)}. \tag{11}$$

(ii) $\mathrm{der}^{-1}\, t$-(v,k,λ) *is defined if and only if* $c^{\mathrm{der}^{-1}}_{t,v,k} \,\big|\, \frac{\lambda}{\Delta\lambda(t,v,k)}$ *where*

$$c^{\mathrm{der}^{-1}}_{t,v,k} := \frac{\Delta\lambda(t+1,v+1,k+1)}{\Delta\lambda(t,v,k)} = \frac{\frac{\binom{k+1}{t+1}}{\gcd\left(\binom{k+1}{t+1},\binom{v+1}{t+1}\right)}}{\gcd\left(\Delta\lambda(t,v,k), \frac{\binom{k+1}{t+1}}{\gcd\left(\binom{k+1}{t+1},\binom{v+1}{t+1}\right)}\right)}. \tag{12}$$

(iii) $\mathrm{res}^{-1}\, t$-(v,k,λ) *is defined if and only if*

(a) $t < k$ *and*

(b) $c^{\mathrm{res}^{-1}}_{t,v,k} \,\big|\, \frac{\lambda}{\Delta\lambda(t,v,k)}$ *where*

$$c^{\mathrm{res}^{-1}}_{t,v,k} = \frac{\Delta\lambda(t+1,v+1,k)\cdot(v+1-k)}{\gcd\big(\Delta\lambda(t+1,v+1,k)\cdot(v+1-k), \Delta\lambda(t,v,k)\cdot(k-t)\big)}. \tag{13}$$

Proof:

(i) The operator $D = \mathrm{red}^{-1}$ is not defined for parameter sets with $t = k$. Hence assume $t < k$. By (4), $\mathrm{der}^{-1}\, t-(v,k,\lambda) = (t+1) - \big(v,k,\lambda\frac{k-t}{v-t}\big)$ is admissible if and only if

$$\begin{aligned}
& \Delta\lambda(t+1,v,k) \;\Big|\; \lambda\frac{k-t}{v-t} \\
\overset{(9)}{\iff}\quad & (v-t)\cdot\frac{k-t}{\gcd(v-t,k-t)}\cdot\frac{\Delta\lambda(t,v,k)}{\gcd\big(\Delta\lambda(t,v,k),\frac{v-t}{\gcd(v-t,k-t)}\big)} \;\Big|\; \lambda\cdot(k-t) \\
\iff\quad & \underbrace{\frac{\frac{v-t}{\gcd(v-t,k-t)}}{\gcd\left(\Delta\lambda(t,v,k),\frac{v-t}{\gcd(v-t,k-t)}\right)}}_{=:c^{\mathrm{red}^{-1}}_{t,v,k}} \;\Big|\; \frac{\lambda}{\Delta\lambda(t,v,k)}.
\end{aligned}$$

(ii) Using (4) again, we find that $\mathrm{der}^{-1}\, t\text{-}(v,k,\lambda) = (t+1)\text{-}(v+1,k+1,\lambda)$ is admissible if and only if

$$\begin{aligned}
&\Delta\lambda(t+1,v+1,k+1) \;\Big|\; \lambda \\
\overset{(10)}{\Longleftrightarrow}\quad & \frac{\Delta\lambda(t+1,v+1,k+1)}{\Delta\lambda(t,v,k)} = \underbrace{\frac{\frac{\binom{k+1}{t+1}}{\gcd\left(\binom{k+1}{t+1},\binom{v+1}{t+1}\right)}}{\gcd\left(\Delta\lambda(t,v,k), \frac{\binom{k+1}{t+1}}{\gcd\left(\binom{k+1}{t+1},\binom{v+1}{t+1}\right)}\right)}}_{=:c^{\mathrm{der}^{-1}}_{t,v,k}} \;\Big|\; \frac{\lambda}{\Delta\lambda(t,v,k)}.
\end{aligned}$$

(iii) If $t = k$, the operator res^{-1} is not defined for $t\text{-}(v,k,\lambda)$. Hence assume $t < k$. By (4), $\mathrm{res}^{-1}\, t\text{-}(v,k,\lambda) = (t+1)\text{-}(v+1,k,\lambda\frac{k-t}{v+1-k})$ is admissible if and only if

$$\begin{aligned}
&\Delta\lambda(t+1,v+1,k) \;\Big|\; \lambda\frac{k-t}{v+1-k} \\
\Longleftrightarrow\quad & \Delta\lambda(t+1,v+1,k)\cdot(v+1-k) \;\Big|\; \frac{\lambda}{\Delta\lambda(t,v,k)}\cdot\Delta\lambda(t,v,k)\cdot(k-t) \\
\Longleftrightarrow\quad & \underbrace{\frac{\Delta\lambda(t+1,v+1,k)\cdot(v+1-k)}{\gcd(\Delta\lambda(t+1,v+1,k)\cdot(v+1-k), \Delta\lambda(t,v,k)\cdot(k-t))}}_{=:c^{\mathrm{res}^{-1}}_{t,v,k}} \;\Big|\; \frac{\lambda}{\Delta\lambda(t,v,k)}.
\end{aligned}$$

□

As the complete design extends under each of the operators (assuming $t < k$ for $D = \mathrm{red}^{-1}$ or $D = \mathrm{res}^{-1}$) we get:

Corollary 3.5

$$c^{D}_{t,v,k} \;\Big|\; \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)} \tag{14}$$

for all $0 \le t \le k \le v$ *and* $D \in \{\mathrm{red}^{-1}, \mathrm{der}^{-1}\,\mathrm{res}^{-1}\}$. *If* $D = \mathrm{red}^{-1}$ *or* $D = \mathrm{res}^{-1}$, *we require* $t < k$.

Example 3.6 (cf. Examples 3.1, 3.3) For which λ does the parameter set $5\text{-}(24,8,\lambda)$ extend under $D \in \{\mathrm{red}^{-1}, \mathrm{der}^{-1}, \mathrm{res}^{-1}\}$? As $t < k$, we get a result in all three cases. Using Proposition 3.4, we compute

(i) $c^{\mathrm{red}^{-1}}_{5,24,8} = \frac{\frac{19}{\gcd(19,3)}}{\gcd\left(1,\frac{19}{\gcd(19,3)}\right)} = 19$, hence $\mathrm{red}^{-1}\, 5\text{-}(24,8,19) = 6\text{-}(24,8,19\cdot\frac{3}{19}) = 6\text{-}(24,8,3)$ is admissible. This is in accordance with $\Delta\lambda(6,24,8) = 3$ (cf. Example 3.3).

(ii) $c^{\mathrm{der}^{-1}}_{5,24,8} = \frac{3}{\gcd(1,3)} = 3$, hence $\mathrm{der}^{-1}\, 5\text{-}(24,8,3) = 6\text{-}(25,9,3)$ is admissible. This is in accordance with $\Delta\lambda(6,25,9) = 3$ (cf. Example 3.3).

(iii) $c^{\mathrm{res}^{-1}}_{5,24,8} = \frac{3\cdot 17}{\gcd(3\cdot 17, 1\cdot 3)} = 17$, hence $\mathrm{res}^{-1}\, 5\text{-}(24,8,17) = 6\text{-}(25,8,3)$ is admissible. This is in accordance with $\Delta\lambda(6,25,8) = 3$ (cf. Example 3.3).

$\diamond$

4 Clans of design parameter sets

In the previous section, we encountered collections of design parameter sets with equal t, v and k and whose set of indices form multiples of a certain number. We call that a clan (recall from (5) that $\Delta\lambda(t,v,k)$ divides $\lambda_{\max}(t,v,k)$):

Definition 4.1 The *clan* of the parameter quadruple (t,v,k,s) with $\Delta\lambda(t,v,k) \mid s \mid \lambda_{\max}(t,v,k)$ is

$$\mathrm{Clan}\big(t,v,k,s\big) = \Big\{t\text{-}(v,k,m\cdot s) \;\Big|\; m \in \mathbb{N},\ 1 \le m \le \frac{\lambda_{\max}(t,v,k)}{s}\Big\},$$

i. e. the set of admissible design parameters for t, v and k whose index is a multiple of s. The *full clan* is $\mathrm{Clan}\big(t,v,k\big) := \mathrm{Clan}\big(t,v,k,\Delta\lambda(t,v,k)\big)$. A clan is *trivial* if it consists of just one element. For a natural number $c \mid \frac{\lambda_{\max}(t,v,k)}{s}$, put

$$c\cdot \mathrm{Clan}(t,v,k,s) := \mathrm{Clan}(t,v,k,cs). \tag{15}$$

For s_1 and s_2 with $\Delta\lambda \mid s_i \mid \lambda_{\max}(t,v,k)$ for $i = 1,2$ we have

$$\mathrm{Clan}\big(t,v,k,s_1\big) \subseteq \mathrm{Clan}\big(t,v,k,s_2\big) \iff s_2 \mid s_1, \tag{16}$$

in which case we call $\mathrm{Clan}\big(t,v,k,s_1\big)$ a *subclan* of $\mathrm{Clan}\big(t,v,k,s_2\big)$. For fixed t, v and k, the ordered set of clans $\mathrm{Clan}\big(t,v,k,s\big)$ where $\Delta\lambda(t,v,k) \mid s \mid \lambda_{\max}(t,v,k)$ is anti-isomorphic to the lattice of divisors of $\frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$. A short notation for $\mathrm{Clan}\big(t,v,k,s\big)$ is

$$t\text{-}(v,k,m\cdot s)_{m \le \frac{\lambda_{\max}(t,v,k)}{s}}. \tag{17}$$

Let us get back to the situation of Proposition 3.4:

Proposition 4.2 *Consider* $\mathrm{Clan}(t,v,k)$*, i. e. the set of admissible design parameter sets of the form* $t\text{-}(v,k,m\cdot\Delta\lambda(t,v,k))$ *with* $1 \le m \le \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$.

(i) For any $D \in \{\mathrm{red}^{-1}, \mathrm{der}^{-1}, \mathrm{res}^{-1}\}$*, the set of elements of* $\mathrm{Clan}\big(t,v,k\big)$ *for which* D *is defined is either empty or forms a subclan* $c^D_{t,v,k}\cdot\mathrm{Clan}\big(t,v,k\big)$ *(recall the notation of (15)) where* $c^D_{t,v,k}$ *is as in (11), (12) or (13). The set is empty if and only if* $t = k$ *and* $D = \mathrm{red}^{-1}$ *or* $D = \mathrm{res}^{-1}$.

(ii) For $D \in \{\mathrm{red}^{-1}, \mathrm{der}^{-1}, \mathrm{res}^{-1}\}$, assume there exists a natural number c such that $D\big(t\text{-}(v,k,c\Delta\lambda(t,v,k))\big)$ is defined. Take c to be minimal, i. e. $c = c^D_{t,v,k}$ as in (i). Then $D(t\text{-}(v,k,c\Delta\lambda(t,v,k)) = t'\text{-}(v',k',\lambda')$ with $\lambda' = \Delta\lambda(t',v',k')$. Therefore the mapping

$$D : c \cdot \mathrm{Clan}(t,v,k) \to \mathrm{Clan}(t',v',k'), \tag{18}$$
$$D\big(t\text{-}(v,k,m \cdot c \cdot \Delta\lambda(t,v,k))\big) = t'\text{-}(v',k',m \cdot \Delta\lambda(t',v',k'))$$

for all natural numbers m with $1 \le m \le \frac{\lambda_{\max}(t,v,k)}{c\cdot\Delta\lambda(t,v,k)}$ is surjective, hence bijective. In other words, the operator D induces a bijection between the subclan $c\cdot\mathrm{Clan}(t,v,k)$ and the full clan $\mathrm{Clan}(t',v',k')$. On the other hand, if c is an integer such that

$$D\big(t\text{-}(v,k,c \cdot \Delta\lambda(t,v,k))\big) = t'\text{-}(v',k',\Delta\lambda(t',v',k'))$$

then c is minimal, i. e. $D\big(t\text{-}(v,k,d \cdot \Delta\lambda(t,v,k))\big)$ is defined for no integer d less than c. Moreover

$$c = \frac{\lambda_{\max}(t,v,k)}{\lambda_{\max}(t',v',k')} \cdot \frac{\Delta\lambda(t',v',k')}{\Delta\lambda(t,v,k)} = \frac{\Delta\lambda(t',v',k')}{\Delta\lambda(t,v,k) \cdot f^D_{t,v,k}} \tag{19}$$

with $f^D_{t,v,k}$ as in (6).

Proof:

(i) Follows from Proposition 3.4. Note that Corollary 3.5 implies $c \mid \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$, hence $c \cdot \mathrm{Clan}(t,v,k)$ is defined.

(ii) In order to avoid confusion, write D^{-1} for the chosen operator of the set $\{\mathrm{red}^{-1}, \mathrm{der}^{-1}, \mathrm{res}^{-1}\}$. We abbreviate $\Delta\lambda = \Delta\lambda(t,v,k)$, $\Delta\lambda' = \Delta\lambda(t',v',k')$, $\lambda_{\max} = \lambda_{\max}(t,v,k)$ and $\lambda'_{\max} = \lambda_{\max}(t',v',k')$. Let $D^{-1}\,t\text{-}(v,k,c\cdot\Delta\lambda) = t'\text{-}(v',k',\lambda')$ with c minimal, i. e. $c = c^D_{t,v,k}$. We are going to prove that $\lambda' = \Delta\lambda'$: If there exists

$$t'\text{-}(v',k',\mu') \in \mathrm{Clan}\big(t',v',k',\Delta\lambda'\big) \setminus D^{-1}\big(\mathrm{Clan}\big(t\text{-}(v,k,c\cdot\Delta\lambda)\big)\big)$$

then $D\,t'\text{-}(v',k',\mu') = t\text{-}(v,k,\mu) \in \mathrm{Clan}(t,v,k,\Delta\lambda)$ implies $\Delta\lambda \mid \mu$. Moreover, the operator D^{-1} is defined for $t\text{-}(v,k,\mu)$ and therefore by (i), $c \mid \frac{\mu}{\Delta\lambda}$ which implies

$$t'\text{-}(v',k',\mu') \in D^{-1}\Big(\mathrm{Clan}\big(t\text{-}(v,k,c\cdot\Delta\lambda)\big)\Big),$$

a contradiction. Hence D^{-1} induces a bijection of the subclan $c \cdot \mathrm{Clan}(t,v,k)$ onto the full clan $\mathrm{Clan}\big(t',v',k'\big)$, namely the map described in (18). We conclude using (6)

$$\big|\mathrm{Clan}\big(t',v',k'\big)\big| = \big|c\cdot\mathrm{Clan}\big(t,v,k\big)\big| \iff \frac{\lambda'_{\max}}{\Delta\lambda'} = \frac{\lambda_{\max}}{c\cdot\Delta\lambda} \iff c = \frac{\lambda_{\max}}{\Delta\lambda}\frac{\Delta\lambda'}{\lambda'_{\max}} = \frac{\Delta\lambda'}{\Delta\lambda\cdot f^D_{t,v,k}}.$$

□

More informally, the previous result tells us that immediate relations in the poset of admissible design parameter sets always come as bijections between a subclan $c \cdot \mathrm{Clan}(t,v,k)$ and a full clan $\mathrm{Clan}(t',v',k')$. Consider our standard example once again:

Example 4.3 (cf. Examples 3.3, 3.6) Having the corresponding values of $\Delta\lambda$ at hand, the computations of Example 3.6 can be simplified using Proposition 4.2:

(i) $c^{\mathrm{red}^{-1}}_{5,24,8} = \frac{\Delta\lambda(6,24,8)}{\Delta\lambda(5,24,8)\cdot f^{\mathrm{red}^{-1}}_{5,24,8}} = \frac{3}{1\cdot\frac{3}{19}} = 19$. Hence $\mathrm{red}^{-1}\, 5\text{-}(24,8,m\cdot 19) = 6\text{-}(24,8,m\cdot 3)$ for all m with $1 \le m \le 51$ induces a bijection between $19\cdot\mathrm{Clan}(5,24,8)$ and $\mathrm{Clan}(6,24,8)$.

(ii) $c^{\mathrm{der}^{-1}}_{5,24,8} = \frac{\Delta\lambda(6,25,9)}{\Delta\lambda(5,24,8)\cdot f^{\mathrm{der}^{-1}}_{5,24,8}} = \frac{3}{1\cdot 1} = 3$. Hence $\mathrm{der}^{-1}\, 5\text{-}(24,8,m\cdot 3) = 6\text{-}(25,9,m\cdot 3)$ for all m with $1 \le m \le 323$ induces a bijection between $3\cdot\mathrm{Clan}(5,24,8)$ and $\mathrm{Clan}(6,25,9)$.

(iii) $c^{\mathrm{res}^{-1}}_{5,24,8} = \frac{\Delta\lambda(6,25,8)}{\Delta\lambda(5,24,8)\cdot f^{\mathrm{res}^{-1}}_{5,24,8}} = \frac{3}{1\cdot\frac{3}{17}} = 17$. Hence $\mathrm{res}^{-1}\, 5\text{-}(24,8,m\cdot 17) = 6\text{-}(25,8,m\cdot 3)$ for all m with $1 \le m \le 57$ induces a bijection between $17\cdot\mathrm{Clan}(5,24,8)$ and $\mathrm{Clan}(6,25,8)$.

In Figure 4, we show the parameters sets which extend once again. Note that the m in that figure is not the m in the previous calculations. ◇

$$\begin{tikzcd}
6\text{-}(24,8,\tfrac{m}{19}\cdot 3)_{\frac{m}{19}\le 51} & 6\text{-}(25,9,\tfrac{m}{3}\cdot 3)_{\frac{m}{3}\le 323} & 6\text{-}(25,8,\tfrac{m}{17}\cdot 3)_{\frac{m}{17}\le 57} \\
& 5\text{-}(24,8,m\cdot 1)_{m\le 969} \arrow[ul, "\substack{\mathrm{red}^{-1}\\ 19|\mathrm{m}}"] \arrow[u, "\substack{\mathrm{der}^{-1}\\ 3|\mathrm{m}}"] \arrow[ur, "\substack{\mathrm{res}^{-1}\\ 17|\mathrm{m}}"'] &
\end{tikzcd}$$

Figure 4: The immediate relations above $5\text{-}(24,8,\lambda)$

Our aim is to describe the relations between admissible parameter sets in terms of the parameters t, v and k only. So far, we have seen that this works for immediate relations. It will turn out shortly that this is true in general. We introduce another relation defined on the set of clans as follows:

Definition 4.4 For integers $t \le k \le v$ and $t' \le k' \le v'$, put

$$\mathrm{Clan}\big(t,v,k\big) \prec \mathrm{Clan}\big(t',v',k'\big) \iff \exists\, \lambda, \lambda' :\; t\text{-}(v,k,\lambda) \le t'\text{-}(v',k',\lambda'), \tag{20}$$

where $\lambda = a\Delta\lambda \le \lambda_{\max}$, $\lambda' = a'\Delta\lambda' \le \lambda'_{\max}$ for some positive integers a and a' and for $\Delta\lambda = \Delta\lambda(t,v,k)$, $\Delta\lambda' = \Delta\lambda(t',v',k')$, $\lambda_{\max} = \lambda_{\max}(t,v,k)$, $\lambda'_{\max} = \lambda_{\max}(t',v',k')$.

Note that the numbers λ and λ' in the previous definition mutually determine each other. Hence we can speak about the minimal λ satisfying (20). The corresponding λ' will then also be minimal.

In case of immediate relations, Proposition 4.2 gives us the number $c^D_{t,v,k}$, which determines the smallest solution and at the same time the subclan of all solutions of (20). We call that number the *associated subclan generator.* We may put it as a subscript to the relation symbol as in the following example:

Example 4.5 (cf. Examples 3.1, 4.3) Proposition 4.2 yields the following inclusions for the clan of 5-$(24, 8, \lambda)$ designs: $\mathrm{Clan}(5, 24, 8) \prec_{19} \mathrm{Clan}(6, 24, 8)$, $\mathrm{Clan}(5, 24, 8) \prec_3 \mathrm{Clan}(6, 25, 9)$ and $\mathrm{Clan}(5, 24, 8) \prec_{17} \mathrm{Clan}(6, 25, 8)$. $\diamond$

The general case of relations between admissible parameter sets is dealt with in the following proposition:

Proposition 4.6 *Assume* $\mathrm{Clan}(t, v, k) \prec \mathrm{Clan}(t', v', k')$, *i. e. there exist nonnegative integers* h, i, j *and* λ, λ' *such that*

$$\mathrm{red}^{-h}\,\mathrm{der}^{-i}\,\mathrm{res}^{-j}\left(t\text{-}(v, k, \lambda)\right) = t'\text{-}(v', k', \lambda') \tag{21}$$

for admissible parameter sets. Abbreviate $\Delta\lambda = \Delta\lambda(t, v, k)$, $\lambda_{\max} = \lambda_{\max}(t, v, k)$ *and similarly* $\Delta\lambda' = \Delta\lambda(t', v', k')$, $\lambda'_{\max} = \lambda_{\max}(t', v', k')$. *Write* $\lambda = c \cdot \Delta\lambda$.

(i) If λ *and* λ' *are minimal with respect to (21) then* $c \mid \frac{\lambda_{\max}}{\Delta\lambda}$ *and* $\lambda' = \Delta\lambda'$. *Hence* $\mathrm{red}^{-h}\,\mathrm{der}^{-i}\,\mathrm{res}^{-j}$ *induces a bijection between the subclan* $c \cdot \mathrm{Clan}(t, v, k)$ *and the full clan* $\mathrm{Clan}(t', v', k')$:

$$\begin{aligned} \mathrm{red}^{-h}\,\mathrm{der}^{-i}\,\mathrm{res}^{-j} :\ c \cdot \mathrm{Clan}(t, v, k) \to \mathrm{Clan}(t', v', k'), \\ \mathrm{red}^{-h}\,\mathrm{der}^{-i}\,\mathrm{res}^{-j}\big(t\text{-}(v, k, m \cdot c \cdot \Delta\lambda)\big) = t'\text{-}(v', k', m \cdot \Delta\lambda') \end{aligned} \tag{22}$$

for all natural numbers m *with* $1 \leq m \leq \frac{\lambda_{\max}}{c\Delta\lambda}$. *Moreover,*

$$c = \frac{\lambda_{\max}}{\lambda'_{\max}} \cdot \frac{\Delta\lambda'}{\Delta\lambda}. \tag{23}$$

On the other hand, if c *is an integer such that*

$$\mathrm{red}^{-h}\,\mathrm{der}^{-i}\,\mathrm{res}^{-j}\left(t\text{-}(v, k, c \cdot \Delta\lambda)\right) = t'\text{-}(v', k', \Delta\lambda')$$

then c *is minimal, i. e.* $\mathrm{red}^{-h}\,\mathrm{der}^{-i}\,\mathrm{res}^{-j}\left(t\text{-}(v, k, d \cdot \Delta\lambda)\right)$ *is defined for no integer* d *less than* c. *We write* $\mathrm{Clan}(t, v, k) \overset{(h,i,j)}{\prec_c} \mathrm{Clan}(t', v', k')$ *and call* (h, i, j) *the* path information *of the relation.*

(ii) *The set of clans is ordered with respect to the relation "$\prec$". The path information is additive and the subclan generator multiplicative with respect to transitivity of that relation.*

(iii) *For* $\mathrm{Clan}(t,v,k) \overset{(h,i,j)}{\prec_c} \mathrm{Clan}(t',v',k')$ *where* $\mathrm{Clan}(t,v,k)$ *is considered fixed, the associated subclan generator together with the path information and* $\mathrm{Clan}(t',v',k')$ *mutually determine each other:*

(a) *(1)* $t' = t+h+i+j$,

(2) $v' = v+i+j$,

(3) $k' = k+i$,

(4) $\Delta\lambda' = c\cdot\Delta\lambda \frac{\binom{k-t}{h}}{\binom{v-t}{h}} \frac{\binom{k-t-h}{j}}{\binom{v+j-k}{j}} = c\cdot\Delta\lambda \frac{\lambda'_{\max}}{\lambda_{\max}}$.

(b) *(1)* $i = k'-k$,

(2) $j = v'-v-i$,

(3) $h = t'-t-i-j$,

(4) $c = \frac{\Delta\lambda'}{\Delta\lambda} \frac{\binom{v-t}{h}}{\binom{k-t}{h}} \frac{\binom{v+j-k}{j}}{\binom{k-t-h}{j}} = \frac{\Delta\lambda'}{\Delta\lambda}\frac{\lambda_{\max}}{\lambda'_{\max}}$.

Proof:

(i) Let us apply the operators $\mathrm{red}^{-h}\mathrm{der}^{-i}\mathrm{res}^{-j}$ one by one in succession, thereby reducing to the case of immediate relations and using Proposition 4.2 for each of the individual steps. Moreover, we assume that minimality forces $\lambda' = \Delta\lambda'$ (this will be justified later). Consider the operators

$$\underbrace{\mathrm{red}^{-1}\circ\cdots\circ\mathrm{red}^{-1}}_{h\text{ times}} \circ \underbrace{\mathrm{der}^{-1}\circ\cdots\circ\mathrm{der}^{-1}}_{i\text{ times}} \circ \underbrace{\mathrm{res}^{-1}\circ\cdots\circ\mathrm{res}^{-1}}_{j\text{ times}}$$

which we are applying one after another from right to left to the parameter set t-$(v,k,c\cdot\Delta\lambda)$. For convenience, put

$$D_\ell = \begin{cases} \mathrm{res}^{-1} & \text{if } 0\le \ell < j, \\ \mathrm{der}^{-1} & \text{if } j \le \ell < i+j, \\ \mathrm{red}^{-1} & \text{if } i+j\le \ell < h+i+j. \end{cases}$$

Applying $D_0, D_1, \ldots, D_{h+i+j-1}$ to t-$(v,k,c\cdot\Delta\lambda)$ we obtain a sequence of admissible parameter sets

$$\Big(t_\ell\text{-}(v_\ell,k_\ell,c_\ell\cdot\Delta\lambda_\ell)\Big)_{\ell=0}^{h+i+j}, \tag{24}$$

with

$$t_0\text{-}(v_0,k_0,c_0\cdot\Delta\lambda_0) = t\text{-}(v,k,c\cdot\Delta\lambda)$$

and

$$t_s\text{-}(v_s, k_s, c_s \cdot \Delta\lambda_s) = t'\text{-}(v', k', 1 \cdot \Delta\lambda(t', v', k'))$$

for $s = h + i + j$ and $\Delta\lambda_\ell = \Delta\lambda(t_\ell, v_\ell, k_\ell)$. The parameter sets are connected by

$$D_\ell\big(t_\ell\text{-}(v_\ell, k_\ell, c_\ell \Delta\lambda_\ell)\big) = t_{\ell+1}\text{-}(v_{\ell+1}, k_{\ell+1}, c_{\ell+1}\Delta\lambda_{\ell+1}) \quad \text{for } \ell = 0, 1, \ldots, h + i + j - 1. \tag{25}$$

We solve these equations for the unknowns c_ℓ backwards, i. e. by solving

$$D_{\ell-1}\big(t_{\ell-1}\text{-}(v_{\ell-1}, k_{\ell-1}, c_{\ell-1} \cdot \Delta\lambda_{\ell-1})\big) = t_\ell\text{-}(v_\ell, k_\ell, c_\ell \cdot \Delta\lambda_\ell)$$

for $\ell = h + i + j, \ldots, 2, 1$. Our initial choice is $c_{h+i+j} = 1$. Using Proposition 4.2 (i), we get

$$c_{\ell-1} = c^{D_{\ell-1}}_{t_{\ell-1}, v_{\ell-1}, k_{\ell-1}} \cdot c_\ell,$$

hence

$$c = c_0 = \prod_{\ell=0}^{h+i+j-1} c^{D_\ell}_{t_\ell, v_\ell, k_\ell}.$$

We have proved that

$$\mathrm{red}^{-h}\mathrm{der}^{-i}\mathrm{res}^{-j}\big(t\text{-}(v, k, c \cdot \Delta\lambda)\big) = t'\text{-}(v', k', \Delta\lambda')$$

holds for that c. As $\Delta\lambda'$ is the smallest solution for λ' we can have, this justifies the assumption made initially. The bijection (22) results from this. Therefore, using (6),

$$\begin{aligned} \big|\mathrm{Clan}(t', v', k')\big| = \big|c \cdot \mathrm{Clan}(t, v, k)\big| &\iff \frac{\lambda'_{\max}}{\Delta\lambda'} = \frac{\lambda_{\max}}{c \cdot \Delta\lambda} \\ \iff c = \frac{\lambda_{\max}}{\Delta\lambda}\frac{\Delta\lambda'}{\lambda'_{\max}} &= \frac{\Delta\lambda'}{\Delta\lambda \cdot \prod_{\ell=0}^{h+i+j-1} f^{D_\ell}_{t_\ell, v_\ell, k_\ell}}. \end{aligned} \tag{26}$$

(ii) We only verify transitivity of "$\prec$": Assume $\mathrm{Clan}(t_1, v_1, k_1) \overset{(h_1,i_1,j_1)}{\prec_{c_1}} \mathrm{Clan}(t_2, v_2, k_2) \overset{(h_2,i_2,j_2)}{\prec_{c_2}}$ $\mathrm{Clan}(t_3, v_3, k_3)$. We then have two bijections of the form (22), induced by

$$\begin{aligned} \mathrm{red}^{-h_1}\,\mathrm{der}^{-i_1}\,\mathrm{res}^{-j_1}\big(t_1\text{-}(v_1, k_1, m_1 \cdot c_1 \cdot \Delta\lambda(t_1, v_1, k_1))\big) &= t_2\text{-}(v_2, k_2, m_1 \cdot \Delta\lambda(t_2, v_2, k_2)), \\ \mathrm{red}^{-h_2}\,\mathrm{der}^{-i_2}\,\mathrm{res}^{-j_2}\big(t_2\text{-}(v_2, k_2, m_2 \cdot c_2 \cdot \Delta\lambda(t_2, v_2, k_2))\big) &= t_3\text{-}(v_3, k_3, m_2 \cdot \Delta\lambda(t_3, v_3, k_3)), \end{aligned}$$

where $1 \leq m_\ell \leq \frac{\lambda_{\max}(t_\ell, v_\ell, k_\ell)}{c_\ell \cdot \Delta\lambda(t_\ell, v_\ell, k_\ell)}$ for $\ell = 1, 2$. Putting $m_1 = c_2$ and $m_2 = 1$ we can combine the equations arriving at

$$\mathrm{red}^{-(h_1+h_2)}\,\mathrm{der}^{-(i_1+i_2)}\,\mathrm{res}^{-(j_1+j_2)}\big(t_1\text{-}(v_1, k_1, c_1c_2 \cdot \Delta\lambda(t_1, v_1, k_1))\big) = t_3\text{-}(v_3, k_3, \Delta\lambda(t_3, v_3, k_3)).$$

By (i), c_1c_2 is minimal in establishing a relation between $\mathrm{Clan}(t_1, v_1, k_1)$ and $\mathrm{Clan}(t_3, v_3, k_3)$. Hence $\mathrm{Clan}(t_1, v_1, k_1) \overset{(h_1+h_2,i_1+i_2,j_1+j_2)}{\prec_{c_1c_2}} \mathrm{Clan}(t_3, v_3, k_3)$, thereby also proving additivity of the path information and multiplicativity of the subclan generator.

(iii) This follows from (26) in the proof of (i). The middle term with the binomial coefficients comes from evaluating the product of the $f^{D_\ell}_{t_\ell,v_\ell,k_\ell}$.

$\square$

Example 4.7 (cf. Example 3.1) Proposition 4.6 yields the following inclusion for parameter sets of the form 5-$(24, 8, \lambda)$ with $323 \mid \lambda$:

$$5\text{-}(24, 8, m \cdot 323) \leq 7\text{-}(25, 8, m \cdot 6) = \mathrm{red}^{-1}\,\mathrm{res}^{-1}\,5\text{-}(24, 8, m \cdot 323)$$

for $1 \leq m \leq 3$. We express this as

$$\mathrm{Clan}(5, 24, 8) \overset{(1,0,1)}{\prec_{323}} \mathrm{Clan}(7, 25, 8).$$

Note that we have $\mathrm{Clan}(5, 24, 8) \overset{(1,0,0)}{\prec_{19}} \mathrm{Clan}(6, 24, 8) \overset{(0,0,1)}{\prec_{17}} \mathrm{Clan}(7, 25, 8)$ and $\mathrm{Clan}(5, 24, 8) \overset{(0,0,1)}{\prec_{17}}$ $\mathrm{Clan}(6, 25, 8) \overset{(1,0,0)}{\prec_{19}} \mathrm{Clan}(7, 25, 8)$ (cf. Fig. 5). $\diamond$

$$\begin{tikzcd}
& 7\text{-}(25, 8, \frac{m}{323} \cdot 6)_{\frac{m}{323} \leq 3} & \\
6\text{-}(25, 8, \frac{m}{17} \cdot 3)_{\frac{m}{17} \leq 57} \arrow[ur, "\substack{\mathrm{red}^{-1} \\ 323|\mathrm{m}}"] & & 6\text{-}(24, 8, \frac{m}{19} \cdot 3)_{\frac{m}{19} \leq 51} \arrow[ul, "\substack{\mathrm{res}^{-1} \\ 323|\mathrm{m}}"'] \\
& 5\text{-}(24, 8, m \cdot 1)_{m \leq 969} \arrow[ul, "\substack{\mathrm{res}^{-1} \\ 17|\mathrm{m}}"] \arrow[ur, "\substack{\mathrm{red}^{-1} \\ 19|\mathrm{m}}"'] &
\end{tikzcd}$$

Figure 5: Some clans above 5-$(24, 8, \lambda)$

Example 4.8 The numbers $c^D_{t,v,k}$ need not be prime:

$$\mathrm{Clan}(5, 32, 8) \overset{(1,0,0)}{\prec_{9}} \mathrm{Clan}(6, 32, 8) \quad (\text{i. e. } c^{\mathrm{red}^{-1}}_{5,32,8} = 9),$$
$$\mathrm{Clan}(8, 41, 10) \overset{(0,0,1)}{\prec_{4}} \mathrm{Clan}(9, 42, 10) \quad (\text{i. e. } c^{\mathrm{res}^{-1}}_{8,41,10} = 4).$$

They even need not be relatively prime:

$$\mathrm{Clan}(5, 32, 8) \overset{(1,0,0)}{\prec_{9}} \mathrm{Clan}(6, 32, 8) \quad (\text{i. e. } c^{\mathrm{red}^{-1}}_{5,32,8} = 9),$$
$$\mathrm{Clan}(5, 32, 8) \overset{(0,1,0)}{\prec_{3}} \mathrm{Clan}(6, 33, 9) \quad (\text{i. e. } c^{\mathrm{der}^{-1}}_{5,32,8} = 3).$$

$\diamond$

Another result about the clans above a given clan is the following:

Proposition 4.9 *If*

$$\mathrm{Clan}(t,v,k) \overset{(h_1,i_1,j_1)}{\prec_{c_1}} \mathrm{Clan}(t_1,v_1,k_1) \quad \textit{and} \quad \mathrm{Clan}(t,v,k) \overset{(h_2,i_2,j_2)}{\prec_{c_2}} \mathrm{Clan}(t_2,v_2,k_2)$$

and $\mathrm{lcm}(c_1,c_2) \neq \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$ *then there is a clan* $\mathrm{Clan}(t_3,v_3,k_3)$ *such that*

$$\mathrm{Clan}(t,v,k) \overset{(\max(h_1,h_2),\max(i_1,i_2),\max(j_1,j_2))}{\prec_{\mathrm{lcm}(c_1,c_2)}} \mathrm{Clan}(t_3,v_3,k_3).$$

The parameters of the clan $\mathrm{Clan}(t_3,v_3,k_3)$ *are determined by Proposition 4.6 (iv)(a).*

Proof: Put $\Delta\lambda = \Delta\lambda(t,v,k)$ and $\lambda_{\max} = \lambda_{\max}(t,v,k)$. Then $t\text{-}(v,k,\mathrm{lcm}(c_1,c_2)\Delta\lambda) \in c_\ell{\cdot}\mathrm{Clan}(t,v,k)$ since $c_\ell \mid \mathrm{lcm}(c_1,c_2)$ for $\ell = 1,2$. Hence

$$\mathrm{red}^{-h_\ell}\mathrm{der}^{-i_\ell}\mathrm{res}^{-j_\ell}t\text{-}(v,k,\mathrm{lcm}(c_1,c_2)\Delta\lambda) \text{ is admissible for } \ell = 1,2.$$

By assumption, $t\text{-}(v,k,\mathrm{lcm}(c_1,c_2)\Delta\lambda)$ is not complete, hence by Lemma 2.13,

$$\mathrm{red}^{-\max(h_1,h_2)}\mathrm{der}^{-\max(i_1,i_2)}\mathrm{res}^{-\max(j_1,j_2)}t\text{-}(v,k,\mathrm{lcm}(c_1,c_2)\Delta\lambda) \text{ is admissible.}$$

It remains to prove that $\mathrm{lcm}(c_1,c_2)$ is the smallest integer $d \leq \frac{\lambda_{\max}}{\Delta\lambda}$ with respect to the property that

$$\mathrm{red}^{-\max(h_1,h_2)}\mathrm{der}^{-\max(i_1,i_2)}\mathrm{res}^{-\max(j_1,j_2)}t\text{-}(v,k,d\Delta\lambda)$$

is defined. Consider such a number d and fix $\ell \in \{1,2\}$. Then $h_\ell \leq \max(h_1,h_2)$, $i_\ell \leq \max(i_1,i_2)$, $j_\ell \leq \max(j_1,j_2)$ imply that $\mathrm{red}^{-h_\ell}\mathrm{der}^{-i_\ell}\mathrm{res}^{-j_\ell}t\text{-}(v,k,d\Delta\lambda)$ is defined (using Lemma 2.11). By minimality of c_ℓ, this implies $c_\ell \mid d$. Thus $\mathrm{lcm}(c_1,c_2) \mid d$ and the statement is proved. □

5 The families of a clan

A clan gives rise to a set of families, generated by the elements of the clan:

Definition 5.1 The *parameterized set of families of* $\mathrm{Clan}(t,v,k)$ is the set

$$\mathcal{F}(t,v,k) = \left\{ \mathcal{F}_m(t,v,k) \;\middle|\; 1 \leq m \leq \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)} \right\} \tag{27}$$

where $\mathcal{F}_m(t,v,k) = \mathrm{Family}(t\text{-}(v,k,m\cdot\Delta\lambda)$. For $(h,i,j) \in \mathbb{N}^3$ satisfying the conditions $h+i+j \leq t$ and $j \leq v-k$, the elements

$$\left\{ \mathrm{red}^h\,\mathrm{der}^i\,\mathrm{res}^j\, t\text{-}(v,k,m\cdot\Delta\lambda) \;\middle|\; 1 \leq m \leq \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)} \right\} \tag{28}$$

are *corresponding members* of $\mathcal{F}(t,v,k)$. The family

$$\mathcal{F}_{\frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}}(t,v,k)$$

is called *complete.* It consists only of complete parameter sets.

Corresponding members of a parameterized set of families form subclans:

Theorem 5.2 *The corresponding members of the set of families $\mathcal{F}(t,v,k)$ form subclans $c \cdot \mathrm{Clan}(t',v',k')$, characterized by the following conditions:*

(i) $0 \leq t' \leq t$,

(ii) $k' \leq v' \leq v$,

(iii) $t' \leq k' \leq k$,

(iv) $k-k' \leq v-v' \leq t-t'$,

(v) $c = \frac{\Delta\lambda(t,v,k)}{\Delta\lambda(t',v',k')}\frac{\lambda'_{\max}}{\lambda_{\max}}$.

The parameter sets of $c \cdot \mathrm{Clan}(t',v',k')$ correspond bijectively to the families $\mathcal{F}(t,v,k)$:

$$t'\text{-}(v',k',m\cdot c\cdot \Delta\lambda(t',v',k')) \in \mathcal{F}_m(t,v,k) \quad \textit{for} \quad 1 \leq m \leq \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}. \tag{29}$$

Proof: We proceed as in the proof of Theorem 2.6. Let t', v', k' and c be integers satisfying (i)-(v). Then $i := k-k'$ and $j := v-v'-(k-k')$ and $h := t-t'-i-j = t-t'-(v-v')$ are nonnegative integers with $h+i+j = t-t' \leq t$ and $j = v-k-(v'-k') \leq v-k$. Hence $\mathrm{red}^h\,\mathrm{der}^i\,\mathrm{res}^j$ is defined for t-$(v,k,\Delta\lambda(t,v,k))$. The equation

$$\mathrm{red}^h\,\mathrm{der}^i\,\mathrm{res}^j\; t\text{-}(v,k,m\Delta\lambda(t,v,k)) = t'\text{-}(v',k',m\lambda) \tag{30}$$

holds for all positive integers $m \leq \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$ and some (fixed) λ. By Proposition 4.6, $\lambda = c\Delta\lambda(t',v',k')$ for the c given in (v). Hence we have a pairing between $\mathrm{Clan}(t,v,k)$ and members of the subclan $c\cdot\mathrm{Clan}(t',v',k')$. By (30), those members are corresponding members in the parameterized set of families $\mathcal{F}(t,v,k)$, with inclusion as in (29).

Conversely, a pairing between $c \cdot \mathrm{Clan}(t',v',k')$ and $\mathrm{Clan}(t,v,k)$ is equivalent to $\mathrm{Clan}(t',v',k') \overset{(h,i,j)}{\prec_c} \mathrm{Clan}(t,v,k)$ for some nonnegative integers (h,i,j). By Proposition 4.6 (with reversed roles of variables), $i = k-k'$, $j = v-v'-i$, $h = t-t'-i-j$ and $c = \frac{\Delta\lambda(t,v,k)}{\Delta\lambda(t',v',k')}\frac{\lambda'_{\max}}{\lambda_{\max}}$. Hence $t = t'+h+i+j \geq t'$ and $v = v'+i+j \geq v'$. Moreover, $k = k'+i \geq k'$. Finally $k-k' = i \leq v-v' = i+j \leq t-t' = h+i+j$. □

Example 5.3 Consider the family of 7-$(25, 8, 6)$, a member of $\mathcal{F}(7, 25, 8)$. The other families are Family(7-$(25, 8, 12)$) and Family(7-$(25, 8, 18)$). We draw the set of families as in Table 1, restricting to the layers of t-design parameter sets with $t \geq 5$. The index is written as a product $m \cdot c \cdot \Delta\lambda$ where c is as in Theorem 5.2. If c is one, the middle factor is omitted. The upper bound for m, i. e. the number $\frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$ is indicated as a subscript of the topmost clan. $\diamond$

7-$(25, 8, m \cdot 6)_{m \leq 3}$		
6-$(25, 8, m \cdot 19 \cdot 3)$	6-$(24, 8, m \cdot 17 \cdot 3)$	
6-$(24, 7, m \cdot 6)$		
5-$(25, 8, m \cdot 19 \cdot 20)$	5-$(24, 8, m \cdot 323 \cdot 1)$	5-$(23, 8, m \cdot 34 \cdot 8)$
5-$(24, 7, m \cdot 19 \cdot 3)$	5-$(23, 7, m \cdot 17 \cdot 3)$	
5-$(23, 6, m \cdot 6)$		

Table 1: The parameterized set of families $\mathcal{F}(7, 25, 8)$

6 Ancestor clans

It may happen that a relation between clans is *trivial* is the sense that the associated subclan generator is one, for instance

$$\mathrm{Clan}(t, v, k) \overset{(h,i,j)}{\prec_1} \mathrm{Clan}(t', v', k')$$

and $h + i + j > 0$. Hence the parameter sets of the two clans correspond bijectively under the operation $\mathrm{der}^{-h} \mathrm{res}^{-i} \mathrm{red}^{-j}$. It turns out that trivial relations are quite frequent, for example all but the first of the chain

$$\begin{aligned} \mathrm{Clan}(5, 24, 8) \overset{(0,1,0)}{\prec_3} \mathrm{Clan}(6, 25, 9) \overset{(0,1,0)}{\prec_1} \mathrm{Clan}(7, 26, 10) \overset{(0,1,0)}{\prec_1} \mathrm{Clan}(8, 27, 11) \\ \overset{(0,1,0)}{\prec_1} \mathrm{Clan}(9, 28, 12) \overset{(0,1,0)}{\prec_1} \mathrm{Clan}(10, 29, 13) \overset{(0,1,0)}{\prec_1} \mathrm{Clan}(11, 30, 14) \\ \overset{(0,1,0)}{\prec_1} \mathrm{Clan}(12, 31, 15) \overset{(0,1,0)}{\prec_1} \mathrm{Clan}(13, 32, 16). \end{aligned}$$

Why does this happen? Of course, the first parameter set of $\mathrm{Clan}(t, v, k)$, i. e. the parameter set t-$(v, k, \Delta\lambda(t, v, k))$ must extend for this to be possible. The other way round, if t-$(v, k, \Delta\lambda(t, v, k))$ is ancestor then no trivial relation can exist above $\mathrm{Clan}(t, v, k)$. Hence we define:

Definition 6.1 A clan $\mathrm{Clan}(t, v, k)$ is called *ancestor clan* if neither red^{-1} nor der^{-1} nor res^{-1} can be applied to the parameter set t-$(v, k, \Delta\lambda(t, v, k))$.

In particular, ancestor clans are always full and never trivial. Ancestor clans are easy to obtain:

Proposition 6.2 *If* $\text{Ancestor}\big(t\text{-}(v,k,\lambda)\big) = t'\text{-}(v',k',\lambda')$ *for some admissible incomplete parameter set* $t\text{-}(v,k,\lambda)$ *then* $\text{Clan}(t',v',k')$ *is ancestor clan.*

Proof: Assume the contrary. Then there are nonnegative integers h, i and j, not all zero, such that $\text{red}^{-h}\,\text{der}^{-i}\,\text{res}^{-j}\,t'\text{-}(v',k',\Delta\lambda')$ is defined, where $\Delta\lambda' := \Delta\lambda(t',v',k')$. Hence with $\lambda' = m\cdot\Delta\lambda'$,

$$\text{red}^{-h}\,\text{der}^{-i}\,\text{res}^{-j}\,t'\text{-}(v',k',m\Delta\lambda') = m\cdot\text{red}^{-h}\,\text{der}^{-i}\,\text{res}^{-j}\,t'\text{-}(v',k',\Delta\lambda')$$

is admissible contrary to the assumption that $t'\text{-}(v',k',\lambda')$ is ancestor parameter set. $\square$

We already have proved most of the following Lemma:

Lemma 6.3 *Let* $t \le k \le v$ *be integers. The following are equivalent:*

(i) $\text{Clan}(t,v,k)$ *is ancestor clan.*

(ii) The parameter set $t\text{-}(v,k,\Delta\lambda(t,v,k))$ *is ancestor parameter set.*

(iii) $\text{Clan}(t,v,k)$ *contains an ancestor parameter set.*

(iv) For every relation $\text{Clan}(t,v,k) \overset{(h,i,j)}{\prec_c} \text{Clan}(t',v',k')$ *with* $h+i+j>0$*, or equivalently* $\text{Clan}(t',v',k') \ne \text{Clan}(t,v,k)$*, the number* c *is different from 1.*

Putting $\lambda := \Delta\lambda(t,v,k)$ in Proposition 6.2 we obtain:

Corollary 6.4 *Every non trivial clan is contained in an unique ancestor clan with associated subclan generator 1.*

As the set of ancestor clans is a subset of the set of all clans, it still forms a poset with respect to the ordering of Definition 4.4. The next property could be called "factorization property":

Proposition 6.5 *Assume* $\text{Clan}(t,v,k) \prec_{c_1} \text{Clan}(t_1,v_1,k_1)$ *and* $\text{Clan}(t,v,k) \prec_{c_2} \text{Clan}(t_2,v_2,k_2)$ *where* $\text{Clan}(t_2,v_2,k_2)$ *is ancestor clan. If* c_1 *divides* c_2 *then* $\text{Clan}(t_1,v_1,k_1) \prec_{c_2/c_1} \text{Clan}(t_2,v_2,k_2)$.

Proof: Write $c_2 = c\cdot c_1$. The assumptions imply

$$t\text{-}(v,k,c_1\cdot\Delta\lambda(t,v,k)) \le t_1\text{-}(v_1,k_1,\Delta\lambda(t_1,v_1,k_1)), \tag{31}$$

$$\text{Ancestor}\big(t\text{-}(v,k,c_2\cdot\Delta\lambda(t,v,k))\big) = t_2\text{-}(v_2,k_2,\Delta\lambda(t_2,v_2,k_2)). \tag{32}$$

"Multiplying" (31) by c, we get $t\text{-}(v,k,c_2\cdot\Delta\lambda(t,v,k)) \le t_1\text{-}(v_1,k_1,c\cdot\Delta\lambda(t_1,v_1,k_1))$. But the ancestor is the largest admissible parameter set above a given one, containing all other with that property, hence together with (32) this implies

$$t_1\text{-}(v_1,k_1,c\cdot\Delta\lambda(t_1,v_1,k_1)) \le t_2\text{-}(v_2,k_2,\Delta\lambda(t_2,v_2,k_2)).$$

Thus $\mathrm{Clan}(t_1,v_1,k_1) \prec_{c_2/c_1} \mathrm{Clan}(t_2,v_2,k_2)$. □

The assumption that $\mathrm{Clan}(t_2,v_2,k_2)$ is ancestor clan is necessary as the following counterexample shows: $\mathrm{Clan}(5,24,8) \prec_3 \mathrm{Clan}(13,32,16)$ and $\mathrm{Clan}(5,24,8) \prec_{57} \mathrm{Clan}(7,24,8)$, but $\mathrm{Clan}(13,32,16) \not\prec \mathrm{Clan}(7,24,8)$.

For ancestor clans, the associated subclan generator determines the successor. The point is that we do not need the path information as in Proposition 4.6 (iii).

Proposition 6.6 *If* $\mathrm{Clan}(t,v,k) \prec_c \mathcal{C}_1$ *for* $c \ne \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$ *and some ancestor clan* $\mathcal{C}_1$*, then* $\mathcal{C}_1$ *is uniquely determined by* c*.*

Proof: Assume

$$\mathrm{Clan}(t,v,k) \overset{(h_1,i_1,j_1)}{\prec_c} \mathcal{C}_1 \quad \text{and} \quad \mathrm{Clan}(t,v,k) \overset{(h_2,i_2,j_2)}{\prec_c} \mathcal{C}_2.$$

for another ancestor clan $\mathcal{C}_2$. Then by Proposition 4.9,

$$\mathrm{Clan}(t,v,k) \overset{(\max(h_1,h_2),\max(i_1,i_2),\max(j_1,j_2))}{\prec_c} \mathcal{C}_3$$

for some clan $\mathcal{C}_3$. Hence

$$\mathrm{Clan}(t,v,k) \overset{(h_1,i_1,j_1)}{\prec_c} \mathcal{C}_1 \overset{(\max(h_1,h_2)-h_1,\max(i_1,i_2)-i_1,\max(j_1,j_2)-j_1)}{\prec_1} \mathcal{C}_3.$$

But $\mathcal{C}_1$ is ancestor clan. Thus Lemma 6.3, (iii), implies $\mathcal{C}_3 = \mathcal{C}_1$. Starting with exchanged roles of $\mathcal{C}_1$ and $\mathcal{C}_2$, we get $\mathcal{C}_2 = \mathcal{C}_3 = \mathcal{C}_1$. In other words, the clan $\mathcal{C}_1$ is uniquely determined by c. □

We conclude

Proposition 6.7 *The poset of ancestor clans above a given clan is finite.*

Proof: Consider an arbitrary clan $\mathrm{Clan}(t,v,k)$. If $\mathrm{Clan}(t,v,k)$ is trivial, no ancestor clan lies above. Hence assume $\mathrm{Clan}(t,v,k)$ is non-trivial. By Corollary 6.4, there is an ancestor clan $\mathrm{Clan}(t_1,v_1,k_1)$ with $\mathrm{Clan}(t,v,k) \prec_1 \mathrm{Clan}(t_1,v_1,k_1)$. By the factorization property (Proposition 6.5), all ancestor clans above $\mathrm{Clan}(t,v,k)$ also lie above $\mathrm{Clan}(t_1,v_1,k_1)$. Every ancestor clan $\mathrm{Clan}(t_i,v_i,k_i)$ above $\mathrm{Clan}(t_1,v_1,k_1)$ satisfies $\mathrm{Clan}(t_1,v_1,k_1) \prec_{c_i} \mathrm{Clan}(t_i,v_i,k_i)$ for some $c_i \mid \frac{\lambda_{\max}(t_1,v_1,k_1)}{\Delta\lambda(t_1,v_1,k_1)} = \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$. Since ancestor clans are never trivial, $c_i < \frac{\lambda_{\max}(t_1,v_1,k_1)}{\Delta\lambda(t_1,v_1,k_1)}$ for all i. By Proposition 6.6, the subclan generator c_i determines the ancestor clan $\mathrm{Clan}(t_i,v_i,k_i)$ uniquely. As there are only finitely many possibilities for c_i, the statement is proved. □

We are able to state the main result about ancestor parameter sets:

Theorem 6.8 *(Klassifikationssatz) The ancestors of incomplete parameter sets of* $\mathrm{Clan}(t, v, k)$ *lie in ancestor clans above that clan. Hence we can classify the set of admissible incomplete t-design parameter sets by the clans containing their ancestors. More precisely, that classification establishes a surjective mapping from the set of admissible incomplete t-design parameter sets to the set of ancestor clans:*

Given an admissible incomplete t-(v, k, λ)*, there is a largest natural number* c *dividing* $\frac{\lambda}{\Delta\lambda(t,v,k)}$ *such that* $\mathrm{Clan}(t, v, k) \prec_c \mathrm{Clan}(t', v', k')$ *for some ancestor clan* $\mathrm{Clan}(t', v', k')$*. Then*

$$\mathrm{Ancestor}\big(t\text{-}(v, k, \lambda)\big) \in \mathrm{Clan}(t', v', k'),$$

hence t-(v, k, λ) *is mapped onto* $\mathrm{Clan}(t', v', k')$*.*

Conversely, a given ancestor clan $\mathrm{Clan}(t', v', k')$ *is the clan of the ancestors of exactly the incomplete design parameter sets contained in families* $\mathcal{F}_m(t', v', k')$ *where* m *is not divisible by any* $c > 1$ *such that* $\mathrm{Clan}(t', v', k') \prec_c \mathrm{Clan}(t'', v'', k'')$*.*

Proof: We first check that the mapping described in the theorem is well defined and really maps onto the clan of the ancestor. Firstly, by Corollary 6.4, the divisor $c = 1$ of $\frac{\lambda}{\Delta\lambda(t,v,k)}$ is always possible, as there always is an ancestor clan $\mathrm{Clan}(t', v', k')$ with $\mathrm{Clan}(t, v, k) \prec_1 \mathrm{Clan}(t', v', k')$ (for this, note that $\mathrm{Clan}(t, v, k)$ is non trivial as it contains an incomplete parameter set). In addition, by Proposition 6.6 that ancestor clan is uniquely determined. Last but not least, Proposition 4.9 implies that the largest c (in the sense of divisibility) really is unique. If $\mathrm{Clan}(t, v, k) \prec_{c_1} \mathrm{Clan}(t_1, v_1, k_1)$ and $\mathrm{Clan}(t, v, k) \prec_{c_2} \mathrm{Clan}(t_2, v_2, k_2)$ for different divisors c_1 and c_2 and ancestor clans $\mathrm{Clan}(t_i, v_i, k_i)$, $i \in \{1, 2\}$ then there is another ancestor clan $\mathrm{Clan}(t_3, v_3, k_3)$ with $\mathrm{Clan}(t, v, k) \prec_{\mathrm{lcm}(c_1,c_2)} \mathrm{Clan}(t_3, v_3, k_3)$. (For this, note that $\mathrm{lcm}(c_1, c_2) \neq \frac{\lambda_{\max}(t,v,k)}{\Delta\lambda(t,v,k)}$ as $\lambda < \lambda_{\max}(t, v, k)$ by assumption.)

In the other direction, we describe the set of parameter sets which are mapped onto a given ancestor clan $\mathrm{Clan}(t', v', k')$. Note that $\mathrm{Clan}(t', v', k') \prec_c \mathrm{Clan}(t'', v'', k'')$ implies that

$$t'\text{-}(v', k', m \cdot \Delta\lambda(t', v', k')) \leq t''\text{-}(v'', k'', \frac{m}{c} \cdot \Delta\lambda(t'', v'', k'')),$$

for all $m \leq \frac{\lambda_{\max}(t',v',k')}{\Delta\lambda(t',v',k')}$ divisible by c. Hence the parameter sets in $\mathcal{F}_m(t', v', k')$ for such m have larger ancestors. $\square$

A few remarks are in order. The importance of Section 4 is that it gives a systematic way to compute all clans including ancestor clans above a given clan. For a given t, v and k, one computes via Proposition 4.2 the subclan generators $c^D_{t,v,k}$ for each of the operations $D \in \{\mathrm{red}^{-1}, \mathrm{der}^{-1}, \mathrm{res}^{-1}\}$, if defined. As long as that number is not $\frac{\lambda_{\max}}{\Delta\lambda}$, one repeats the process after replacing t, v and

k by t', v' and k' where $D\big(t\text{-}(v,k,c\Delta\lambda)\big) = t'\text{-}(v',k',\Delta\lambda')$. The ancestor clans resulting from that computation are stored. Theorem 6.8 allows to classify admissible parameter sets by mapping them to the appropriate ancestor clan. In practice, it turns out that the number of ancestor clans is reasonably small.

Let us get back to the ancestors of the parameter sets 5-$(24, 8, \lambda)$ discussed previously in Example 3.1:

Example 6.9 The relationships between ancestor clans above Clan$(5, 24, 8)$ are the following. Note that Clan$(5, 24, 8)$ itself is ancestor clan.

$$\begin{array}{lll}
\mathrm{Clan}(5,24,8) \overset{(0,8,0)}{\prec_3} \mathrm{Clan}(13,32,16) & \overset{(0,2,2)}{\prec_{17}} & \mathrm{Clan}(17,36,18), \\
\mathrm{Clan}(5,24,8) \overset{(0,0,1)}{\prec_{17}} \quad \mathrm{Clan}(6,25,8) & \overset{(1,0,0)}{\prec_{19}} & \mathrm{Clan}(7,25,8), \\
\mathrm{Clan}(5,24,8) \overset{(1,0,0)}{\prec_{19}} \quad \mathrm{Clan}(6,24,8) & \overset{(1,8,0)}{\prec_3} & \mathrm{Clan}(15,32,16).
\end{array}$$

It is time to draw a picture, thereby discovering more relations between these ancestor clans (cf. Fig. 6). ◇

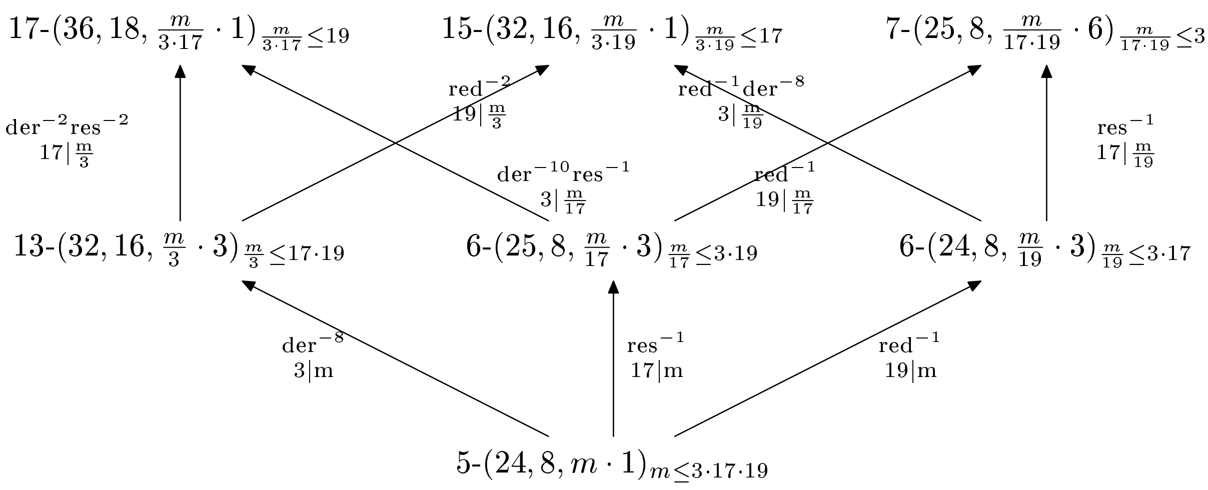


Figure 6: The ancestor clans above Clan$(5, 24, 8)$

7 Ancestor clans of t-designs with large t

Let us get back to the main goal of this paper, which is the classification of known t-designs with $t \geq 5$. We are referring to a list of around 7000 parameter sets of such designs, each of which has been constructed explicitly (at the time of writing this article, which is Spring 2001). Most of these designs have been constructed by researchers in Bayreuth, Germany (we refer to [2]), but the list includes also designs constructed elsewhere. In Table 2, we present the ancestor clans of these t-designs. The ancestor clans are denoted in the form t-$(v, k, m \cdot \Delta\lambda(t, v, k))$ where m varies between 1 and $\lambda_{\max}(t, v, k)/\Delta\lambda(t, v, k)$, as indicated in the subscript. We cannot show detailed information about the families, except that we indicate the number of realizable families and the number of realizable parameter sets for each clan (a family is realizable if it contains is at least one realizable parameter set). Interestingly, we can classify the parameter sets by 80 ancestor clans.

8 Acknowledgments

I thank Prof. R. Laue for the inspiring idea of an ancestor parameter set ("Großvater"). Theorem 2.6 and Theorem 2.14 are due to him (I have presented a slightly modified proof of the second theorem). Prof. Laue also presented to me a list of around 7000 parameter sets of existing t-designs with $t \geq 5$, and the need to classify these of course triggered this whole investigation.

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ancestor clan	# f.	# d.
5-$(12, 6, m \cdot 1)_{m \leq 7}$	2	2
5-$(16, 8, m \cdot 5)_{m \leq 33}$	1	1
5-$(19, 6, m \cdot 2)_{m \leq 7}$	2	2
5-$(24, 8, m \cdot 1)_{m \leq 969}$	284	284
5-$(24, 12, m \cdot 6)_{m \leq 8398}$	2921	2921
5-$(28, 10, m \cdot 7)_{m \leq 4807}$	10	10
5-$(30, 7, m \cdot 30)_{m \leq 10}$	1	1
5-$(32, 8, m \cdot 5)_{m \leq 585}$	17	17
5-$(33, 6, m \cdot 4)_{m \leq 7}$	3	3
5-$(54, 6, m \cdot 1)_{m \leq 49}$	1	1
6-$(14, 7, m \cdot 4)_{m \leq 2}$	1	4
6-$(22, 7, m \cdot 4)_{m \leq 4}$	1	1
6-$(24, 8, m \cdot 3)_{m \leq 51}$	16	16
6-$(25, 8, m \cdot 3)_{m \leq 57}$	18	54
6-$(28, 10, m \cdot 35)_{m \leq 209}$	31	124
6-$(30, 7, m \cdot 12)_{m \leq 2}$	1	4
6-$(33, 8, m \cdot 3)_{m \leq 117}$	7	7
6-$(38, 7, m \cdot 4)_{m \leq 8}$	1	1
6-$(40, 10, m \cdot 2)_{m \leq 23188}$	9	36
7-$(16, 8, m \cdot 3)_{m \leq 3}$	1	5
7-$(20, 10, m \cdot 2)_{m \leq 143}$	56	276
7-$(24, 12, m \cdot 14)_{m \leq 442}$	170	677
7-$(25, 8, m \cdot 6)_{m \leq 3}$	1	4
7-$(32, 16, m \cdot 55)_{m \leq 37145}$	1	1
7-$(34, 8, m \cdot 3)_{m \leq 9}$	2	3
7-$(40, 10, m \cdot 4)_{m \leq 1364}$	8	80
7-$(41, 10, m \cdot 8)_{m \leq 748}$	8	32
8-$(24, 12, m \cdot 70)_{m \leq 26}$	4	35
8-$(31, 10, m \cdot 1)_{m \leq 253}$	109	555
8-$(34, 12, m \cdot 10)_{m \leq 1495}$	4	16
8-$(37, 12, m \cdot 21)_{m \leq 1131}$	32	320
8-$(42, 10, m \cdot 3)_{m \leq 187}$	3	12
8-$(52, 10, m \cdot 2)_{m \leq 473}$	2	2
9-$(20, 10, m \cdot 1)_{m \leq 11}$	5	37
9-$(31, 10, m \cdot 2)_{m \leq 11}$	5	52
9-$(32, 16, m \cdot 33)_{m \leq 7429}$	1	1
9-$(36, 12, m \cdot 15)_{m \leq 195}$	25	211
9-$(42, 10, m \cdot 3)_{m \leq 11}$	2	2
9-$(50, 12, m \cdot 20)_{m \leq 533}$	11	11
10-$(36, 12, m \cdot 5)_{m \leq 65}$	13	167
10-$(37, 12, m \cdot 9)_{m \leq 39}$	7	71
11-$(24, 12, m \cdot 1)_{m \leq 13}$	6	123
11-$(30, 15, m \cdot 6)_{m \leq 646}$	152	1210
11-$(36, 18, m \cdot 220)_{m \leq 2185}$	1	1
11-$(37, 12, m \cdot 2)_{m \leq 13}$	4	81
11-$(45, 15, m \cdot 22)_{m \leq 2108}$	2	8
12-$(46, 16, m \cdot 44)_{m \leq 1054}$	1	4
13-$(30, 15, m \cdot 4)_{m \leq 34}$	8	191
13-$(32, 16, m \cdot 3)_{m \leq 323}$	144	2902
13-$(45, 15, m \cdot 4)_{m \leq 124}$	3	30
14-$(30, 15, m \cdot 8)_{m \leq 2}$	1	55
15-$(32, 16, m \cdot 1)_{m \leq 17}$	8	418
15-$(49, 16, m \cdot 2)_{m \leq 17}$	1	1
16-$(74, 18, m \cdot 3)_{m \leq 551}$	1	1
17-$(36, 18, m \cdot 1)_{m \leq 19}$	9	330
19-$(44, 22, m \cdot 20)_{m \leq 115}$	4	13
21-$(44, 22, m \cdot 1)_{m \leq 23}$	11	90
21-$(48, 24, m \cdot 195)_{m \leq 15}$	2	20
21-$(96, 24, m \cdot 5)_{m \leq 13505}$	3	12
23-$(48, 24, m \cdot 5)_{m \leq 5}$	1	14
23-$(52, 26, m \cdot 42)_{m \leq 87}$	3	24
25-$(52, 26, m \cdot 9)_{m \leq 3}$	1	1
25-$(56, 28, m \cdot 5)_{m \leq 899}$	3	12
26-$(85, 28, m \cdot 1)_{m \leq 1711}$	15	15
27-$(56, 28, m \cdot 1)_{m \leq 29}$	6	11
28-$(60, 30, m \cdot 8)_{m \leq 62}$	2	20
29-$(60, 30, m \cdot 1)_{m \leq 31}$	14	36
30-$(62, 31, m \cdot 16)_{m \leq 2}$	1	9
35-$(72, 36, m \cdot 1)_{m \leq 37}$	2	2
39-$(80, 40, m \cdot 1)_{m \leq 41}$	1	1
41-$(84, 42, m \cdot 1)_{m \leq 43}$	13	15
45-$(92, 46, m \cdot 1)_{m \leq 47}$	1	1
57-$(116, 58, m \cdot 1)_{m \leq 59}$	2	2
59-$(120, 60, m \cdot 1)_{m \leq 61}$	2	2
65-$(132, 66, m \cdot 1)_{m \leq 67}$	1	1
77-$(156, 78, m \cdot 1)_{m \leq 79}$	1	1
101-$(204, 102, m \cdot 1)_{m \leq 103}$	1	1
125-$(252, 126, m \cdot 1)_{m \leq 127}$	1	1
161-$(324, 162, m \cdot 1)_{m \leq 163}$	1	1
237-$(476, 238, m \cdot 1)_{m \leq 239}$	2	2

Table 2: Ancestor clans of existing t-designs with $t \geq 5$ (# f. = # families, # d. = # design parameter sets)