Graphical t-Designs via Polynomial Kramer-Mesner Matrices

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Abstract

Kramer-Mesner matrices have been used as a powerful tool to construct t-designs. In this paper we construct Kramer-Mesner matrices for fixed values of k and t in which the entries are polynomials in n the number of vertices of the underlying graph. From this we obtain an elementary proof that with a few exceptions $S_n^{[2]}$ is a maximal subgroup of $S_{\binom{n}{2}}$ or $A_{\binom{n}{2}}$. We also show that there are only finitely many graphical incomplete t-(v, k, λ) designs for fixed values of $2 \leq t$ and k at least in the cases $k = t + 1$, $t = 2$, and $2 \leq t < k \leq 6$. All graphical t-designs are determined by the program DISCRETA for various small parameters. Most parameter sets are new for graphical designs, some also for general simple t-designs. The largest value of t for which graphical designs were found is $t = 5$. Some of the smaller designs which are block transitive are drawn as graphs.

Key words: Graphical t-designs, maximal subgroups of symmetric groups, polynomial Kramer-Mesner matrices

1 Introduction

Consider the action of the symmetric group S_n on the set $V = \binom{X}{2}$ where $X = \{1, 2, \cdots, n\}$. This defines an embedding of S_n into $S_{\binom{n}{2}}$ with image group $S_n^{[2]}$. Any subset K of V can be considered as a labelled graph with edge set

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K and vertex set X. The orbits of $S_n^{[2]}$ on 2^V are just the isomorphism classes of graphs and thus such an orbit can be described by drawing an unlabelled graph. If a collection of these orbits for a fixed k forms a t-design then this design is called a graphical design (on n points).

We give appropriate collections of isomorphism classes of graphs such that the resulting collections of subsets of V form a t-design. The condition for a design that each t-subset of V be contained in the same number λ of blocks in this context means that each labelled graph T with t edges and vertex set X occurs exactly λ times as a subgraph in the selected collection of labelled graphs. We don't allow a multiple occurrence of the same graph, i.e. we only consider simple designs. Usually, we also are not interested in the complete design consisting of all graphs with k edges or the empty design. So, without explicit mentioning we mean by a design an incomplete non-empty design.

There are a few papers considering this special kind of designs. It seems that W. O. Alltop [1] first constructed graphical 2-$(\binom{n}{2}, k, \lambda)$-designs, where K is the set of edges of a cycle of length k and $n = 2 \times k - 3$.

Our approach uses Kramer-Mesner matrices which are named after an influential paper [17]. In fact, the method is a systematic version of finding a tactical decomposition of a structure by means of the automorphism group of the structure. Similar approaches have a long history which we do not want to trace back here. We only report that in the early 70's M. H. Klin [15] already used Kramer-Mesner matrices (as they are called now) with polynomial entries to describe graphical 2- and 3-designs for showing that $S_n^{[2]}$ is a maximal subgroup of $S_{\binom{n}{2}}$ with only a few small exceptions. A new version of this result is part of the present paper. Besides the note [14] with the finiteness theorem which was published in a local journal not known even in the former USSR his results never were published. A similar approach of polynomial matrices for another class of groups can also be found in the report of L. H. M. E. Driessen [11]. Moreover, Driessen gave graphical 3-$(10, 5, 6)$, 3-$(10, 4, 1)$, and 3-$(15, 5, 30)$ designs. In [5], Chee also shows 3 Kramer-Mesner matrices with polynomial entries for graphical t-designs. Only, there is no indication in that paper how these matrices can be found.

Important contributions to the theory of graphical designs were made by L. G. Chouinard II, E. S. Kramer, and D.L. Kreher [9], who determined the graphical designs for $\lambda = 1, 2$.

P. J. Cameron and C. E. Praeger [4, example 1.4], obtain a flag-transitive graphical 2-$(78, 15, \lambda)$ design from the Petersen graph, enlarged by 3 isolated vertices. Two further examples for graphical designs have been published by E. S. Kramer [16] and Y. M. Chee [7]. Kramer's design has parameters 3-$(21, 5, 3)$ and Chee's design has parameters 3-$(28, 5, 30)$. Additional material has been reported by Y. M. Chee [8].

Starting from Alltop's approach [1] we show how polynomial Kramer-Mesner matrices can be obtained. We use our program DISCRETA to construct such matrices for smaller cases. The graphical designs obtained for 2-$(v, 3, \lambda)$ and 3-$(v, 4, \lambda)$ are used in an elementary way to determine all overgroups of $S_n^{[2]}$ in $S_{\binom{n}{2}}$ following Klin [14,15]. Nowadays this result can also be obtained from the classification of finite simple groups [20].

From the polynomial Kramer-Mesner matrices we obtain some results on graphical designs for infinitely many parameter sets. It was known that there exist only finitely many graphical designs with parameter sets of type 2-$(v, 3, \lambda)$, 2-$(v, 4, \lambda)$, 3-$(v, 4, \lambda)$, and none for 4-$(v, 5, \lambda)$, [8], [6]. We show that there is also no graphical 5-$(v, 6, \lambda)$ design and, more generally, for each k there exist only finitely many graphical t-(v, k, λ) designs in each of the following cases:

- $k = t + 1$,
- $t = 2$,
- $2 \leq t < k \leq 6$.

The proof leads to conjecture that such a finiteness result might hold for all fixed pairs (t, k). Then, there would exist only sporadic graphical designs for these parameters. We thus determine many such sporadic designs and, surprisingly, find examples even for $t = 5$. The results are reported in two tables in Section 6.

2 Preliminaries

If X is any finite set and k a natural number then

$$\binom{X}{k} = \{K \mid K \subset X, |K| = k\},$$
$$X^k = \{(x_1, x_2, \ldots, x_k) \mid x_i \in X \text{ for all } i\}.$$

A tuple $(x_1, x_2, \ldots, x_k) \in X^k$ is injective if all components are pairwise different. We denote

$$X_{\text{inj}}^k = \{(x_1, x_2, \ldots, x_k) \mid (x_1, x_2, \ldots, x_k) \text{ injective tuple from } X^k\}.$$

Let G be a group acting on X. In particular, there is always the full symmetric group $S(X)$ on X. We denote the image of $x \in X$ under $g \in G$ by x^g. Then

G also acts on $\binom{X}{k}$ by $K^g = \{x^g \mid x \in K\}$ for $K \in \binom{X}{k}$, and on X^k by $(x_1, x_2, \ldots, x_k)^g = (x_1^g, x_2^g, \ldots, x_k^g)$ for $(x_1, x_2, \ldots, x_k) \in X^k$. We denote by $G^{[2]}$ the permutation group induced by G on $\binom{X}{2}$. Especially we will use this for $G = S_n$, the symmetric group on $\{1, 2, \ldots, n\}$. A_n is the alternating group on $\{1, 2, \ldots, n\}$.

The set X^k_{inj} is closed under G, since each $g \in G$ acts as a bijective function on X. Therefore we have a mapping

$$\varphi : X^k_{\text{inj}} \longrightarrow \binom{X}{k} : (x_1, x_2, \ldots, x_k) \mapsto \{x_1, x_2, \ldots, x_k\}$$

which commutes with the action of G. We denote by $k_{()}$-orbits and $k_{\{\}}$-orbits the respective orbits of G on X^k_{inj} and $\binom{X}{k}$. Usually $k_{\{\}}$-orbits are also denoted as k-orbits. We will follow this convention when no misunderstanding is likely.

A $k_{()}$-orbit Φ of injective k-tuples is *totally symmetric*, if with each $(x_1, x_2, \ldots, x_k) \in \Phi$ also each $(x_{1^\pi}, x_{2^\pi}, \ldots, x_{k^\pi}) \in \Phi$ for each permutation π of the k components. Thus, for a totally symmetric $k_{()}$-orbit Φ we have $|\Phi| = k! \cdot |\varphi(\Phi)|$.

For a $K \in \binom{X}{k}$ the setwise stabilizer of K in G is

$$N_G(K) = \{g \mid g \in G, \text{for all } x \in K \ x^g \in K\}.$$

It is clear that $N_{S_n}(K) = Aut(\Gamma)$, where $Aut(\Gamma)$ is the automorphism group of a graph $\Gamma = (X, K)$. We also call this the *normalizer* of K in G. Then the pointwise stabilizer of K in G is

$$C_G(K) = \{g \mid g \in G, \text{for all } x \in K \ x^g = x\}.$$

We also call this the *centralizer* of K in G. The centralizer of K is just the kernel of the permutation representation of $N_G(K)$ defined by the restriction of the action to K. Φ is totally symmetric if and only if for each $K \in \Phi$ $N_G(K)/C_G(K) \cong S_k$. G is called $k_{()}$-transitive (or simply k-transitive), if there exists only one $k_{()}$-orbit. If G is $k_{()}$-transitive but not $(k+1)_{()}$-transitive then G is called *exactly* $k_{()}$-transitive. G is called k-homogeneous if there exists only one $k_{\{\}}$-orbit.

Remark: Do not mix the notation of an exactly $k_{()}$-transitive permutation group with that of a sharply $k_{()}$-transitive permutation group (the latter is defined, e. g., in [10], p.210).

For $t \geq 2$ a simple t-(v, k, λ) design D defined on a set V with $|V| = v$ is a set of blocks $B \subset \binom{V}{k}$ such that each $T \in \binom{V}{t}$ is contained in exactly λ blocks of D. The maximal value of λ is $\binom{v-t}{k-t}$. A design with this value of λ is called the *complete* design, otherwise it is an incomplete design. The trivial design consists of no blocks. If λ is just half of the maximal value, the design is called a *halving* of the complete design. Generally, the k-subsets not in a design D also form a design, called the *complementary* design $\bar{D}$ of D.

For the rest of the paper we denote $X = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, and $V = \binom{X}{2}$. A subset $E \subset V$ is considered as the set of edges of an undirected graph with vertex set X. A graphical t-(v, k, λ) design D is a simple t-(v, k, λ) design admitting $S_n^{[2]}$ as a group of automorphisms.

3 Polynomial Kramer-Mesner matrices

A group A of automorphisms of a design D, or more exactly a subgroup A of the automorphism group of a t-(v, k, λ) design D acting on the point set V has orbits on the set of blocks of the design. Thus, the design is a collection of A-orbits on $\binom{V}{k}$. If a t-subset T is contained in $m(T, K^A)$ elements of the orbit K^A of $K \in \binom{V}{k}$ then also each T^a for $a \in A$ is contained in the same number of elements of that orbit. For a collection of A-orbits on $\binom{V}{k}$ one only has to test for a set of representatives of the t-orbits if they appear in exactly λ elements of the selected orbits. This observation has been formalized by Kramer and Mesner [17].

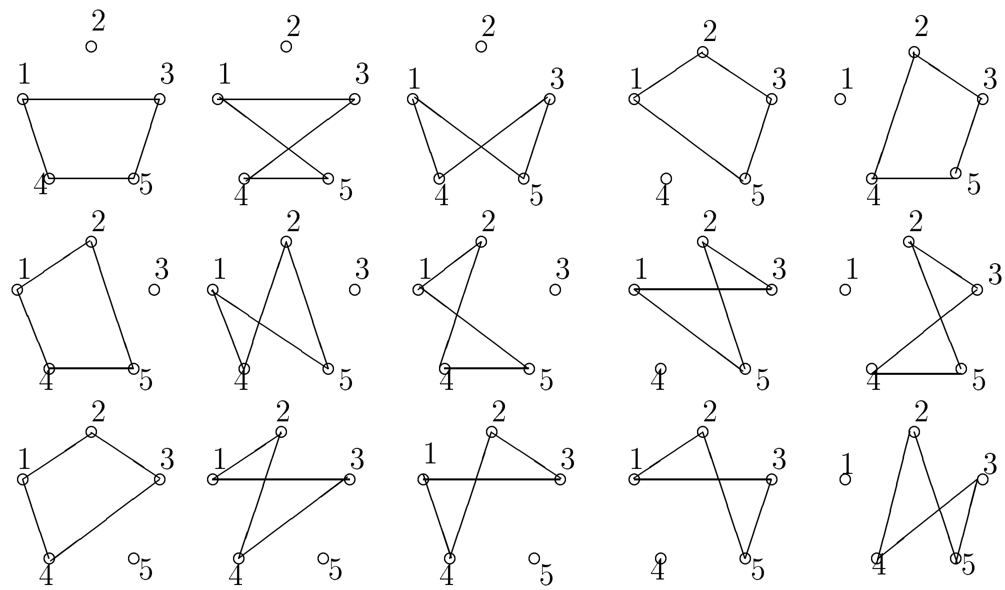
Theorem 1 *(Kramer, Mesner 1976) A t-(v, k, λ) design exists with $A \leq S_v$ as a group of automorphisms if and only if there is a $\{0, 1\}$-solution vector u to the diophantine system of equations*

$$\sum_j m(T_i, K_j^A) u_j = \lambda$$

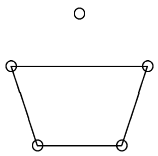
where the T_i and the K_j run through a system of representatives of the t-orbits and k-orbits of A, respectively.

If only one k-orbit already forms a design, i.e. u has only one nonzero entry, the design is called block-transitive. In this paper we consider the case of $A = S_n^{[2]}$ where $V = \binom{X}{2}$ for $X = \{1, 2, \ldots, n\}$. These designs are graphical, since they can be visualized by graphs. We give an example of a block-transitive graphical 2-(10,4,2) design, which is taken from [9].

Each pair of edges of the complete labelled graph on 5 vertices appears exactly twice in the following 15-element set of graphs with 4 vertices.

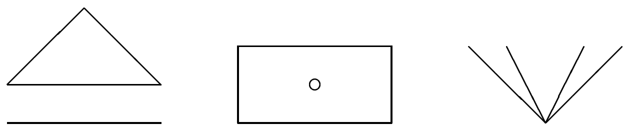


Under the action of $S_5^{[2]}$ on the set of labelled graphs on 5 vertices the shown graphs form one orbit. This is therefore just an isomorphism class of graphs which can be represented by an unlabelled graph.



If a graphical design is formed by more than one orbit we have to draw a graph for each orbit. A first example of this kind has been attributed to R. M. Wilson by Kramer and Mesner [17]. It has the parameters 3-(10,4,1) and is shown in the next figure. We denote this design by $D(3,4)$.

The 3-$(10,4,1)$ design $D(3,4)$



The design $D(3,4)$ and some other cases appear in [3] and [21] as a nice illustration of the notion of a design. Further examples can be found in Handbook of Combinatorial designs [8] or be constructed by DISCRETA. We have listed many such designs in Section 6. These usually have so many orbits that drawing the graphs is not feasible.

Instead of computing a Kramer-Mesner matrix for each $(\binom{n}{2}, k, t)$ we now introduce Kramer-Mesner matrices whose entries are polynomials in n. So for each k and t we have only one matrix covering all infinitely many $n \geq 2k$.

The basic tool goes back to Alltop (1966). We give a slightly different version here. For $T \in \binom{V}{t}$ and $B \in \binom{V}{k}$ we use

$$m(T, B^G) = \{(T, K) | K \in B^G, T \subset K\}$$

and

$$m'(T^G, B) = \{(S, B) | S \in T^G, S \subset B\}.$$

Counting the pairs in

$$\{(S, K) | S \in T^G, K \in B^G, S \subset K\}$$

in two ways and using $|B^G| \cdot |N_G(B)| = |G|$ one obtains the following relation.

Lemma 2 (Alltop [1])

$$m(T, B^G) \times |N_G(B)| = m'(T^G, B) \times |N_G(T)|.$$

Consider an orbit K^G where $G = S_n^{[2]}$. Then K is a k-subset of $V = \binom{X}{2}$, where $X = \{1, 2, \ldots, n\}$. Since there are usually several vertices not incident with any edge in K, we denote by

$$\mathrm{supp}(K) = \{i | \exists \{i, j\} \in K\}$$

the set of vertices which are incident with an edge of K. The following lemma is evident.

Lemma 3 *With the above notation*

$$\mathrm{Aut}(K) = A \times S(X \setminus \mathrm{supp}(K)),$$

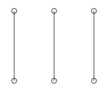
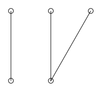
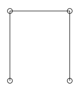
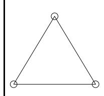
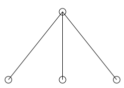
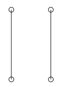
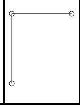
where A is a permutation group on $\mathrm{supp}(K)$ *which does not depend on X.*

So, for graphical designs in Alltop's Lemma we can replace $N_G(B)$ and $N_G(T)$ by these automorphism groups of the corresponding graphs.

In order to compute the Kramer-Mesner matrix for graphical t-(v, k, λ) designs we can first determine for each pair of isomorphism types of graphs K

and T with k and t edges, respectively, the number of embeddings of T in K, i.e. $m'(T^G, B)$. This can be done for a fixed n, say $n = |\mathrm{supp}(K)|$. Then from Alltop's Lemma one obtains the entries $m(T, B^G)$ for all values of n. As an alternative one can start from a Kramer-Mesner matrix M for a fixed n, as computed by DISCRETA, and then use Alltop's Lemma to first determine $m'(T^G, B)$ in each case, before proceeding as above. The resulting Kramer-Mesner matrix has entries which are polynomials in n with rational coefficients and which take only integer values for all integers n. We computed the polynomial Kramer-Mesner matrices given in the appendix using DISCRETA without any hand calculation.

We demonstrate this approach with a first polynomial Kramer-Mesner matrix which we need in the next section. Let $t = 2$ and $k = 3$. We form the polynomial Kramer-Mesner matrix for $n \geq 6$.

		K_1	K_2	K_3	K_4	K_5
						
T_1		$\frac{(n-4)(n-5)}{2}$	$4(n-4)$	4	0	0
T_2		0	$\frac{(n-3)(n-4)}{2}$	$2(n-3)$	1	$n-3$

As an example we compute $m(T_2, K_2^G)$, where $G = S_n^{[2]}$. We have that

$$|\mathrm{supp}(T_2)| = 3, \quad \mathrm{Aut}(T_2) \cong S_2 \times S_{n-3} .$$

So, $|\mathrm{Aut}(T_2)| = 2 \cdot (n-3)!$. Similarly, $|\mathrm{Aut}(K_2)| = 4 \cdot (n-5)!$. T_2 can be embedded in only one way into K_2, hence $m'(T_2^G, K_2) = 1$. Alltop's formula yields

$$m(T_2, K_2^G) = m'(T_2^G, K_2) \cdot \frac{|N_G(T_2)|}{|N_G(K_2)|} = 1 \cdot \frac{2 \cdot (n-3)!}{4 \cdot (n-5)!} = \frac{(n-3)(n-4)}{2} .$$

In our tables we use the shorter notation $[n-3]_2$ of falling factorials for the term $(n-3)(n-4)$. For $n = 8$ the columns K_1 and K_3 have in each row the constant sum 10. So, taking the orbits K_1^G and K_3^G results in a graphical 2-(28,3,10) design. For each of the 2^5 subsets of $\{K_1, K_2, \ldots, K_5\}$ one obtains a system of Diophantine equations whose solutions give all graphical 2-$(\binom{n}{2}, 3, \lambda)$ designs for $n \geq 6$. The smaller values of n can be handled directly. In the next section all solutions for this case are listed. For $t=3$, $k=4$ there is only one graphical design, the 3-(10,4,1) design $D(3,4)$ we showed above.

4 Overgroups of $S_n^{[2]}$ in $S_{\binom{n}{2}}$

With the help of graphical 2-designs we show

Theorem 4 (Klin (1970, 1974) [14,15]) *Let $n \geq 9$ or $n = 7$. Then*

(1) $S_n^{[2]} \underset{\max}{<} S_{\binom{n}{2}}$ if n is odd,

(2) $S_n^{[2]} \underset{\max}{<} A_{\binom{n}{2}}$ if n is even.

Corollary 5 *For $n \geq 9$ or $n = 7$ all graphical t-designs are pairwise nonisomorphic.*

The exceptional cases are covered by the following result.

Theorem 6 (Klin (1974) [15]) *Up to complementary designs there exist exactly the following nontrivial graphical 2-$(\binom{n}{2}, 3, \lambda)$ and 3-$(\binom{n}{2}, 4, \lambda)$ designs D with automorphism group $\mathrm{Aut}(D)$, obtained from the polynomial Kramer-Mesner matrices in the first two tables in the appendix.*

n	D	$t-(v,k,\lambda)$	$\mathrm{Aut}(D)$	$k_{()}-$trans.
5	$D(3,4)$	$3-(10,4,1)$	$\mathrm{Aut}(S_6)$	3
5	K_3	$2-(10,3,4)$	S_6	2
6	$K_1 \cup K_4$	$2-(15,3,1)$	$\mathrm{PSL}(4,2) \cong A_8$	2
8	$K_1 \cup K_4 \cup K_5$	$2-(28,3,6)$	$S_8^{[2]}$	1
8	$K_1 \cup K_3$	$2-(28,3,10)$	$\mathrm{Sp}(6,2)$	2
11	$K_1 \cup K_3 \cup K_4 \cup K_5$	$2-(55,3,25)$	$S_{11}^{[2]}$	1

The listed automorphism groups and $S_4 \times S_2$ in case $n=4$ are the only proper overgroups of $S_n^{[2]}$ in $S_{\binom{n}{2}}$, besides $A_{\binom{n}{2}}$ if n is even.

Remark 7 *Chee 1991, see [8], also independently found that only the graphical designs in Theorem 6 are possible for 2-$(v,3,\lambda)$ and 3-$(v,4,\lambda)$.*

Remark 8 *The 2-(10,3,4) design is a halving of the complete design. This design is isomorphic to its complement.*

It is easy to deduce the Corollary 5 from Theorem 4. Since in the listed cases $S_n^{[2]}$ is a maximal subgroup of $S_{\binom{n}{2}}$ or $A_{\binom{n}{2}}$, any incomplete graphical t-$(\binom{n}{2}, k, \lambda)$ design must have $S_n^{[2]}$ as its full automorphism group. Any $g \in S_{\binom{n}{2}}$ mapping one incomplete graphical t-$(\binom{n}{2}, k, \lambda)$ design D_1 onto another D_2 must map $\mathrm{Aut}(D_1)$ onto $\mathrm{Aut}(D_2)$ via conjugation. But we have $\mathrm{Aut}(D_1) = S_n^{[2]} = \mathrm{Aut}(D_2)$ such that g normalizes $S_n^{[2]}$. Again, since $S_n^{[2]}$ is a maximal subgroup it coincides with its normalizer such that $g \in S_n^{[2]}$. Thus, g is an automorphism of D_1 and $D_1 = D_2$. This argument is just a special case of a theorem in Schmalz [22], see also Laue [19].

The proof of Theorem 4 is reduced to an existence problem of graphical t-designs using the following general observation.

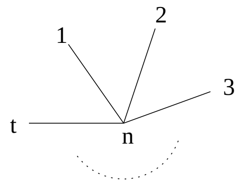
Lemma 9 *Let G be a permutation group of a set X with a totally symmetric $k_{()}$-orbit Φ for some $k \geq 2$. Suppose that $G < A < S(X)$ and that A is exactly $(k-1)_{()}$-transitive. Then for some appropriate λ there exists an incomplete $(k-1)$-$(|X|, k, \lambda)$ design admitting A as a group of automorphisms.*

PROOF. By assumption we have a totally symmetric $k_{()}$-orbit $\Phi = (x_1, x_2, \ldots, x_k)^G$. Then the $k_{()}$-orbit $(x_1, x_2, \ldots, x_k)^A$ will also be totally symmetric, since $N_G(\{x_1, x_2, \ldots, x_k\})/C_G(\{x_1, x_2, \ldots, x_k\})$ is embedded into $N_A(\{x_1, x_2, \ldots, x_k\})/C_A(\{x_1, x_2, \ldots, x_k\})$ and thus both factor groups are isomorphic to S_k. We also know that A is not $k_{()}$-transitive. So there exists a k-tuple $(y_1, y_2, \ldots, y_k) \notin (x_1, x_2, \ldots, x_k)^A$. We will show that then also $\{y_1, y_2, \ldots, y_k\} \notin \{x_1, x_2, \ldots, x_k\}^A$. Otherwise for some $a \in A$ we would have $\{y_1, y_2, \ldots, y_k\}^a = \{x_1, x_2, \ldots, x_k\}$ and $(x_1, x_2, \ldots, x_k)$ could be mapped by permuting the components onto $(y_1^a, y_2^a, \ldots, y_k^a)$. But such a permutation is already contained in $N_A(\{x_1, x_2, \ldots, x_k\})$ such that $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ would be in the same orbit of A. This contradicts to our selection of $(y_1, y_2, \ldots, y_k)$.

So we can form a Kramer-Mesner matrix for the $(k-1)_{\{\}}$-orbits and the $k_{\{\}}$-orbits of A. Since A is $(k-1)_{\{\}}$-transitive, this matrix has just one row. As we have just shown there are at least two $k_{\{\}}$-orbits resulting in at least two different columns of the Kramer-Mesner matrix. Taking any column of the matrix then yields an incomplete $(k-1)$-$(|X|, k, \lambda)$ design with A as a group of automorphisms. $\square$

Assuming Theorem 6 to be true we can now prove Theorem 4.

PROOF. For $n-1 \geq t \geq 2$ the group $S_n^{[2]}$ always has a totally symmetric $t_{()}$-orbit, i.e. the orbit of the star graph



$$(\{1,n\},\{2,n\},\ldots,\{t,n\})^{S_n^{[2]}}.$$

So we can take $S_n^{[2]}$ as the group G in Lemma 9. For $n > 4$ also $S_n^{[2]}$ has two $2_{()}$-orbits besides the diagonal and is the full automorphism group of the graphs representing these $2_{()}$-orbits by the Whitney-Jung theorem, see [23,13,12]. Therefore, each overgroup A of $S_n^{[2]}$ in $S_{\binom{n}{2}}$ has to fuse these orbits.

Each overgroup A thus is $2_{()}$-transitive. If A is not $3_{()}$-transitive then by Lemma 9 A must be an automorphism group of an incomplete graphical 2-$(\binom{n}{2}, 3, \lambda)$ design.

By Theorem 6 such a design does not exist if n is not equal to 5, 6, 8, 11. For $n = 11$ there exist 2-$(\binom{n}{2}, 3, \lambda)$ designs. But these have $S_n^{[2]}$ as their full automorphism group. Thus, also in this case A cannot exist.

So A must be $3_{()}$-transitive. Assume that A is not $4_{()}$-transitive. Then again by the lemma we must find an incomplete graphical 3-$(\binom{n}{2}, 4, \lambda)$ design. From Theorem 6 we see that for $n = 7$ and $n > 9$ such an overgroup A does not exist.

So we are left with the case of $S_n^{[2]} < A < S_{\binom{n}{2}}$ and A $4_{()}$-transitive, where the Bochert-Manning theorem [24] yields $n \leq 7$. So only $n = 7$ remains which can be handled directly. $\square$

For the proof of Theorem 6 the polynomial Kramer-Mesner matrices are used. In the cases when there is no proper overgroup a consideration of some intersection numbers of the designs shows that different orbits of $S_{[2]}$ can not be merged into one orbit of a suitable overgroup. In other cases DISCRETA at least shows that the overgroups are groups of automorphisms of these t-designs. It again has to be shown like before that they are full automorphism groups. The transitivity results follow from Lemma 9.

5 Finiteness theorems for graphical designs

The polynomial Kramer-Mesner matrix can be analysed to deduce nonexistence results for infinitely many parameter sets of graphical designs.

Theorem 10 *There exist only finitely many graphical nontrivial incomplete*

t-(v, k, λ) designs for any fixed pair (t, k) in each of the following cases:

- $k = t + 1$,
- $t = 2$,
- $2 \le t < k \le 6$.

PROOF. The cases 2-$(v, 3, \lambda)$, 2-$(v, 4, \lambda)$, 3-$(v, 4, \lambda)$, 4-$(v, 5, \lambda)$ are reported in [8]. We thus concentrate on the remaining cases.

In the proof of this theorem we always consider the matrix in its transposed version, due to the printing format. That makes it easier to compare the steps of the proof with the polynomial Kramer-Mesner matrices displayed in the tables of the appendix. We always refer to the corresponding tables in our proof. The columns are indexed by the t-set orbits and the rows are indexed by the k-set orbits. A graphical t-(v, k, λ) design corresponds to a selection of rows such that in each column the sum over the entries of the selected rows is exactly λ. We thus refer to these rows as forming the designs and compose the design out of selected rows. For any such design D all rows which do not belong to D also form a design, the complementary design $\bar{D}$. This is clear, since the total sum in each column is $\binom{v-t}{k-t}$. We thus may assume without loss of generality that one selected row belongs to D.

For $t = 3$ and $k = 5$ we obtain a 26×5 matrix, see Table 5. Let D be a graphical 3-$(v, 5, \lambda)$ design, corresponding to some rows of the matrix. So, for some fixed value of n the entries in each column from those rows sum up to the same value of λ. We claim that for a big enough number of points n such a common sum value λ is possible only if either D or $\bar{D}$ is the complete design. To see this we first notice that each column contains exactly one entry which is a polynomial of degree 4 in n. A polynomial of degree 4 grows faster than the sum of all polynomials of degree $4-1$ or less of any other column. Assume that a row with a polynomial entry of degree 4 belongs to D. Let n be big enough such that the same sum λ can only be obtained from selected entries in every column if also in each column the polynomial of degree 4 contributes to this sum. Thus, all rows containing an entry of degree 4 have to belong to D. Then the last column has only entries of degree 2 or less not yet in rows of design D. Now we in turn consider the design $\bar{D}$. This design has a value of λ which must be a sum of these remaining entries of degree 2 or less in the last column. So, also in each other column this value of λ for $\bar{D}$ for large enough n must not come from any polynomial of a degree higher than 2. Thus all rows containing any polynomial of degree higher than 2 have to belong to D. Assigning these rows to D then results in row candidates for $\bar{D}$ which have constant entries in the last column. Repeating our argument assigns all rows with non constant entries to D such that D has to be the complete design.

The same arguments can be applied in the cases $k = 6$, $t = 3$, Table 7, and $k = 6$, $t = 4$, Table 8. Also here each polynomial of maximal degree appears just once in each column. As before we may assume that the corresponding rows belong to the design D. After eliminating all rows which contain an entry of maximal degree, in the remaining matrix each entry in the last column is a polynomial of a degree less than the maximal degree in its row. So, recursively, we have to assign each row to D if n is large enough. The assumed design is therefore the complete design.

Next consider the case $t = 2$, see Table 1 to Table 4. We have a fixed k but may choose the number of points n as large as needed. There are only two columns to be considered. By Alltop's Lemma a matrix entry is 0 if and only if the t-subset is not contained in any k-subset of the considered k-orbit. If the graph consisting of two incident edges and isolated points cannot be embedded in a graph with k edges then the k edges are pairwise non-incident. The isomorphism type of this graph is thus uniquely determined and we have just one 0 entry in the first column, located in the last row, say. So, with only one exception, whenever the other graph with two edges can be embedded into a k-edge graph, also this graph is embeddable.

The graph consisting of two non-incident edges has one isolated point less than the graph consisting of two incident edges. Therefore, by Alltop's Lemma, for each k-graph, i.e. graph with exactly k edges, the polynomial entries of the general Kramer-Mesner matrix in the corresponding row have degrees differing by 1 unless an entry is 0. Thus, for each row the entry in the first column either has a degree strictly greater than the degree in the second column unless the entry in the first column is 0. Since the sum of all polynomials in both columns must be the same, this last row must contain the only entry of maximal degree in the second column. Now assume a graphical 2-(v, k, λ) design D containing the graph of the last row. We argue as above that for large enough n the design D must contain all graphs of the other rows with a polynomial entry of this degree. By our observation that the second column always has an entry of a degree 1 less than the first column with the only exeption of 0-polynomials, we obtain that all entries of second maximal degree in the second column belong to rows which are assigned to D. Thus all rows with an entry of that degree must also belong to D. We can now proceed by induction to see that D must be the complete design.

The case $k = t + 1$ is quite similar to the above, compare Table 6. We assume that $k > 4$, since the smaller cases are already known. The graph $I(t)$, consisting of t pairwise non intersecting edges, is contained only in such graphs with $t + 1$ edges that result from adding one edge (in the examples this graph always corresponds to the last column). The new edge may again be isolated, or intersect just one edge, or intersect exactly two edges. So, the polynomial Kramer-Mesner matrix contains non-zero entries in the column of $I(t)$ only in the rows of these three graphs with $t+1$ edges. Because $t+1>3$ each of these graphs has an isolated edge.

Let us look at the entries of the matrix in this column in the corresponding 3 rows.

By Alltop's Lemma the degree of the polynomials is just the difference between the sizes of the supports of the graphs considered. So, in the case of an additional isolated edge the support grows by two points such that we get a polynomial of degree 2. If the new edge intersects only one existing edge just one point is added and if the new edge intersects two edges no point is added. So we get a linear polynomial and a constant entry in these cases, respectively.

Both of the last two rows also contain an entry of degree 2 in other columns. The graphs of these columns are easily obtained by removing an isolated edge from the graph on $t+1$ edges considered. We assume a graphical design D containing the graph corresponding to the entry of degree 2 in the last column. For large enough n all rows with an entry of degree 2 correspond to D. So, then all rows belong to D which have a non-zero entry in the last column. Then $\bar{D}$ must be the trivial design and D is the complete design. $\Box$

We conjecture that indeed the stronger result than finiteness holds for the first case of the theorem, i. e. for $4 \leq t$ no graphical t-$(v, t+1, \lambda)$ designs exist. This is known to hold for $t=4$ [8], [6], and we add the case $t=5$ here.

Theorem 11 *There exists no incomplete graphical* 5-$(v, 6, \lambda)$ *design.*

PROOF. We refer to the polynomial Kramer-Mesner matrix of which the first 7 columns are displayed in Table 6 in the appendix, see further comments in Section 7. As in the preceding proof we start with the observation that the last column 26 has only three entries not zero, namely $1/2(n-10)(n-11),\ 10(n-10),\ 40$.

An incomplete design D must not contain all graphs corresponding to the rows with these entries. Since either D or $\bar{D}$ contains the graph of the row with entry $1/2(n-10)(n-11)$, we assume w. l. o. g. that D does not contain this graph. So, λ has the form $\lambda = a40 + b10(n-10)$, where $a, b \in \{0, 1\}$ and not both values are 0. We have to consider three cases for λ. In each case we first examine the first column. There results a linear combination of the non-zero entries which has to add up to the chosen value of λ in each case.

I. $\lambda = 40$. There we have to solve

$$x_1 + x_2 1/2(n-4)(n-5) + x_3 2(n-4) + x_4 2(n-4) = 40$$

with $x_i \in \{0,1\}$. If $x_2 = 1$ then $n \leq 13$. Checking these values leaves one case $n = 10$, but then $40 = \lambda_{max}$ such that D is the complete design. Thus $x_2 = 0$. Division by two yields that $x_1 = 0$ and $(x_3 + x_4)(n-4) = 20$. We are left with the cases $n = 14$ and $n = 24$. Consider column number 5. There $n = 24$ is not possible, but for $n = 14$ there exist the row combinations $(4,12)$ and $(4,8,16)$. Since row 14 has a non-zero entry in column 5, in both cases row 14 must not belong to D. Looking at column number 4 we can exclude as well row 13. Now column 3 has no solutions for $\lambda = 40$ left.

II. $\lambda = 10(n-10)$. There we have to solve

$$x_1 + x_2 1/2(n-4)(n-5) + x_3 2(n-4) + x_4 2(n-4) = 10n - 100$$

with $x_i \in \{0,1\}$. For $x_2 = 1$ we get a quadratic equation

$$n^2 - 29n + 220 + 2y = 0$$

where y is some natural number. This equation has no solution in natural numbers, since $n^2 - 29n + 220$ has a positive minimal value as a polynomial in n. For $x_2 = 0$ we also have $x_1 = 0$ and we can divide by 2.

$$(x_3 + x_4)(n-4) = 5n - 50.$$

So, 5 divides $n - 4$. Substituting $n = 5z + 4$ yields

$$(x_3 + x_4)5z = 25z - 30,$$

and

$$(x_3 + x_4)z = 5z - 6.$$

There is only one solution, namely $z = 2, x_3 = x_4 = 1$. This means $n = 14, \lambda = 40$, but $\lambda = 40$ had been ruled out in the previous case.

III. $\lambda = 10(n-10) + 40$. There we have to solve

$$x_1 + x_2 1/2(n-4)(n-5) + x_3 2(n-4) + x_4 2(n-4) = 10(n-10) + 40$$

with $x_i \in \{0,1\}$.

For $x_2 = 1$ we get a quadratic equation

$$n^2 - 29n + 140 + 2x_1 + 4(x_3 + x_4)(n-4) = 0.$$

The equation $n^2 - 29n + 140 + 2x_1 = 0$ has no integer solutions. Therefore $x_3 + x_4 \in \{1, 2\}$.

For $x_3 + x_4 = 1$ the formula reduces to

$$n^2 - 25n + 124 + 2x_1 = 0.$$

This equation has integer solutions only for $x_1 = 1$ and then $n = 18$ or $n = 7$. Thus, either $\lambda = 120$ or $\lambda = 10$. Column 7 has no solution in both cases.

If $x_3 + x_4 = 2$ the formula reduces to

$$n^2 - 21n + 108 + 2x_1 = 0.$$

This equation has the solutions $n = 10,\ n = 11$ for $x_1 = 1$ and $n = 9$, $n = 12$ for $x_1 = 0$ in natural numbers. Thus $\lambda \in \{120, 130, 140, 150\}$. Column 4 then has too small entries with these values of n to reach these values of λ.

For $x_2 = 0$ we get the linear equation

$$x_1 + 2(x_3 + x_4)(n - 4) = 10(n - 10) + 40.$$

Then $x_1 = 0$ and the equation reduces to

$$(5 - (x_3 + x_4))n = 30 - 4(x_3 + x_4),$$

which has neither for $x_3 + x_4 = 1$ nor for $x_3 + x_4 = 2$ any solution in natural numbers. $\square$

From the polynomial Kramer-Mesner matrices for $t = 2$ and $k = 5$ one obtains easily all isomorphism types of block transitive graphical 2-$(v, 5, \lambda)$ designs. The parameter sets for $v \leq 36$ are already listed in the preceding table. In addition there exist 2 isomorphism types for 2-$(171, 5, 131040)$ and one isomorphism type for 2-$(741, 5, 29216880)$.

From this we see that finiteness does not mean small numbers in our theorems. In fact, for non-block transitive graphical 2-$(55, 5, \lambda)$ designs there exist solutions for all $\lambda \in \{4512, 4540, 4652, 5212, 5352, 5576, 5912, 7312, 7256, 7900, 7942, 8572, 8712, 8600, \ldots\}$. Thus we expect quite many solutions for larger v.

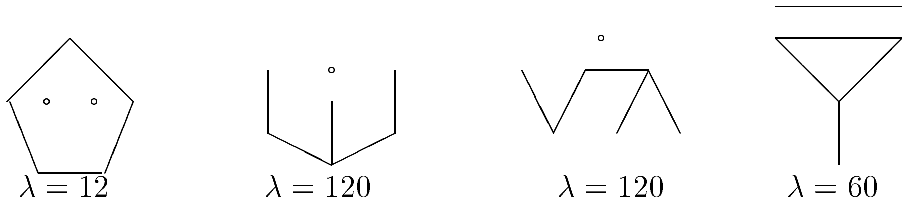
6 Sporadic graphical designs

In this section we present a huge number of various graphical designs which were found using DISCRETA. The following tables give parameter sets for graphical designs found. The headline for each column in the tables shows the number of isomorphism types. Each table entry in this column lists the values of λ for which this number of isomorphism types exists. Generally, the dots indicate that the computation was not pursued to larger values of λ. The rows correspond to the values of n and k. In some cases there exist block transitive designs which are shown in a separate row. For some values of k there are so many designs that we restricted only to block transitive designs. For $k=10$ there are no block transitive graphical designs on 7 points. For 2-$(28, 9, \lambda)$ there exist many further values of λ with not block transitive graphical designs.

2-Designs

	1	2	3	4	5	6	7	≥ 8
2-(21,5,λ) block transitive	12, 60	120	-	-	-	-	-	-
2-(21,5,λ)	7, 12, 19, 22, 34, 60, 64, 79, 94	35, 47, 50, 52, 55, 57, 62, 69, 70, 72, 77, 84, 89, 95,	67, 82, 100, ...					
2-(21,6,λ)	13, 30, 48, 50, 55, 61, 68, 70, ...	38, 45, 51, 58, 63			60			
2-(21,7,λ)	63, 78, 105, ...	42		84				
2-(21,8,λ) block transitive	-	84	-	-	336, 672	168	-	-
2-(21,9,λ) block transitive	12, 54, 108	216	-	72, 864	-	-	432	-
2-(21,10,λ)	99, 162	189, ...	180					
2-(28,5,λ)	60, 140		100					160, 200, 240, 260, 300, 340, 360, ...
2-(28,6,λ)	25, 40, 50, 70, 80, 90	65, 100, ...						
2-(28,7,λ)	16	140	156	182	198			
2-(28,8,λ)	70, 210, ...							
2-(28,9,λ) block transitive	40, 160, 640	-	320	-	-	-	480	960, 1920, 3840
2-(36,5,λ) block transitive	60, 80, 240, 480, 720	-	-	-	-	-	-	-
2-(36,5,λ)	140, 164, 244, ...	180, 224			240			
2-(36,6,λ) block transitive	20, 45, 240, 540, 720, 2160, 4320	-	120, 1080	-	-	-	-	-
2-(36,7,λ)	246, 372, 462,	210, 336		546			456 ...	420

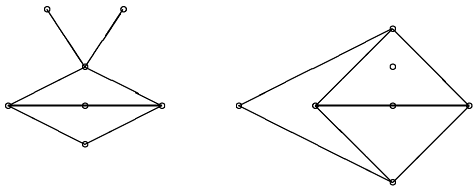
The block transitive 2-$(21, 5, \lambda)$ designs for $\lambda = 12,\ 60,\ 120$ are depicted in the next figure. The first is Alltop's example.



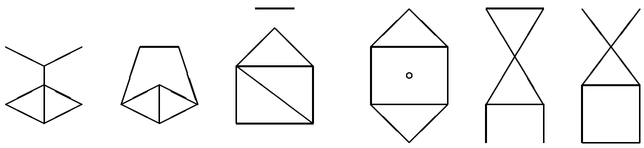
$\lambda = 12$ $\lambda = 120$ $\lambda = 120$ $\lambda = 60$

We also show the block transitive 2-$(21, 8, \lambda)$ designs:

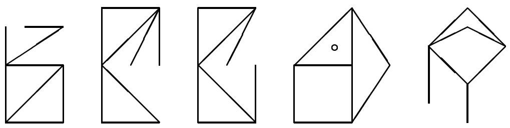
$\lambda = 84$



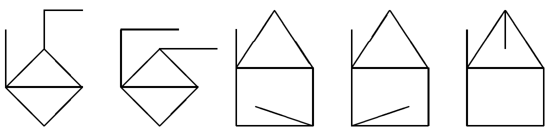
$\lambda = 168$



$\lambda = 336$



$\lambda = 672$



t-Designs, $t \geq 3$

	1	2	3	4	5	6	7	8	≥ 9
3-(21,5,λ)	3 [16], 30, 33	48	39, 75		69				
3-(21,6,λ)		68,108			100	136			128,140,148, 156,160,168, 176,180,188, 196,200,...
3-(21,7,λ)		105							120,210,225, 315, ...
3-(21,8,λ)									168, 252, 336, 420, ...
3-(28,5,λ)	30 [7], 150								
3-(28,6,λ)	80,180	120		220					240,260,...
3-(28,7,λ)	225	210			245				240,275,...
3-(28,8,λ)	168							378	672 ...
3-(28,9,λ)	280, ...								
3-(36,5,λ)	180		270 ...						
4-(28,6,λ)					132				
5-(28,7,λ)	93	-	-	-	-	-	-	-	-
5-(28,8,λ)									756, 791, 840, 875 ...
5-(36,7,λ)									165 ...

It should be noticed that at least the 5-designs have new parameter sets even when compared to general simple t-designs, see Kreher's list in the CRC handbook of combinatorial designs [18]. There exists only one graphical 5-$(28, 7, \lambda)$ design and then $\lambda = 93$. It seems remarkable that this parameter set can also be obtained by deriving the newly found 8-(31,10,93) designs three times [2]. The designs are found as solutions to the Kramer-Mesner diophantine system of linear equations. For example, there are more than 10 million solutions for graphical 5-$(28, 8, 756)$ designs. Similar to the proof of Corollary 5 it is not necessary to do pairwise isomorphism testing, since the overgroups of $S_n^{[2]}$ are known in these cases.

For example, $S_8^{[2]}$ is not a maximal subgroup of S_{28}. There exists an overgroup isomorphic to $Sp(6, 2)$ in S_{28}. This group is transitive on 2-sets and thus has a Kramer-Mesner matrix consisting of only one row. Taking any single k-orbit then results in a block-transitive 2-design. Any combination of them gives another 2-design.

DISCRETA shows that this overgroup is admitted as an automorphism group of only a few t-designs with the listed parameters. These parameters are 2-$(28, 5, \lambda)$ where λ is a sum of the numbers $160, 200, 640, 800, 800$, and 2-$(28, 6, \lambda)$ where λ is a sum of the numbers $40, 50$, $80, 200$, $480, 900$, 1200, 2400, 2400, 7200, 2-$(28, 7, \lambda)$ where λ is a sum of the numbers $16, 420, 560, 672, 1120, 1120, 1680, 1680, 4032, 5760, 5040, 6720, 10080, 10080, 10080$, and 2-$(28, 8, \lambda)$ where λ is a sum of the numbers $70, 448, 1120, 1120, \cdots$. So, some of the graphical designs with $k = 5$ have the larger automorphism group $Sp(6, 2)$. The number of isomorphism types could only be in doubt for 2-$(28, 5, \lambda)$ where $\lambda \in \{160, 200\}$. There are actually 10 solutions for $\lambda = 160$ and 19 solutions for $\lambda = 200$. In both cases DISCRETA computed their intersection numbers and showed that the designs are pairwise non-isomorphic. For $k \geq 6$ the number of isomorphism types in the table is unchanged if the automorphism group is actually larger than $S_8^{[2]}$. It is remarkable that for the small $\lambda = 16$ in case $k = 7$ the design turns out to be block-transitive under $Sp(6, 2)$.

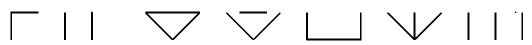
The situation is different for $t \geq 3$, for then t-designs with automorphism group $Sp(6, 2)$ have values of λ larger than those listed in our table of graphical designs. Therefore, for $t \geq 3$ all graphical designs with these parameters have $S_8^{[2]}$ as full automorphism group and are pairwise non-isomorphic.

We are pleased to acknowledge the anonymous referees for numerous helpful remarks and improvements.

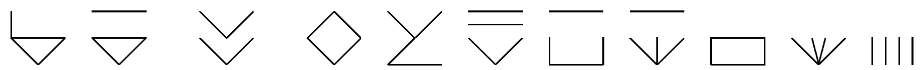
7 Appendix: Polynomial Kramer-Mesner matrices

The graphs with 2 to 5 edges are used in the order shown in the following pictures.

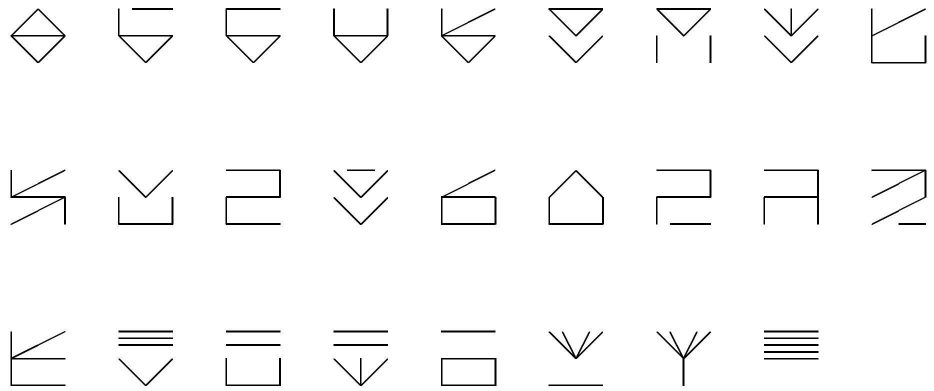
K_1^2 to K_2^2, K_1^3 to K_5^3



K_1^4 to K_{11}^4



K_1^5 to K_{26}^5



In the following tables $[\mathrm{n}]_i = n(n-1)\dots(n-i+1)$ are the falling factorials of n of length i, S = support, A_S = automorphism group restricted to the support. In the case $t = 5,\ k = 6$ we have displayed only the first 7 out of 26 columns and also only the first 23 out of 68 rows, since the first 7 columns have non-zero entries only in these rows. Our proof that no incomplete non-trivial graphical 5-$(v, 6, \lambda)$ design exists needs only this part of the matrix and one additional column, as described in the proof.

Table 1

$3\backslash 2$			1	2
	$\lvert S\rvert$		3	4
		$\lvert A_S\rvert$	6	4
1	3	6	1	0
2	5	4	$1/2[n-3]_2$	$4[n-4]$
3	4	2	$2[n-3]$	4
4	4	6	$1[n-3]$	0
5	6	48	0	$1/2[n-4]_2$

Table 2

$4\backslash 2$			1	2
	$\lvert S\rvert$		3	4
		$\lvert A_S\rvert$	2	12
1	4	2	$5[n-3]$	4
2	5	12	$1/2[n-3]_2$	$2[n-4]$
3	6	8	$1/2[n-3]_3$	$4[n-4]_2$
4	5	2	$3[n-3]_2$	$12[n-4]$
5	5	2	$4[n-3]_2$	$8[n-4]$
6	7	16	$1/8[n-3]_4$	$5/2[n-4]_3$
7	6	4	$1[n-3]_3$	$8[n-4]_2$
8	6	12	$1/2[n-3]_3$	$2[n-4]_2$
9	4	8	$1[n-3]$	2
10	5	24	$1/2[n-3]_2$	0
11	8	384	0	$1/8[n-4]_4$

Table 3

$5\backslash 2$			1	2
	$\lvert S\rvert$		3	4
		$\lvert A_S\rvert$	4	4
1	4	4	$4[n-3]$	4
2	6	4	$5/2[n-3]_3$	$10[n-4]_2$
3	5	2	$6[n-3]_2$	$16[n-4]$
4	5	2	$7[n-3]_2$	$12[n-4]$
5	5	4	$4[n-3]_2$	$4[n-4]$
6	6	12	$2/3[n-3]_3$	$4[n-4]_2$
7	7	48	$1/8[n-3]_4$	$7/6[n-4]_3$
8	7	12	$2/3[n-3]_4$	$4[n-4]_3$
9	6	2	$5[n-3]_3$	$20[n-4]_2$
10	6	8	$3/2[n-3]_3$	$4[n-4]_2$
11	7	4	$3/2[n-3]_4$	$14[n-4]_3$
12	6	2	$4[n-3]_3$	$24[n-4]_2$
13	8	16	$1/4[n-3]_5$	$4[n-4]_4$
14	5	2	$6[n-3]_2$	$16[n-4]$
15	5	10	$1[n-3]_2$	$4[n-4]$
16	7	4	$3/2[n-3]_4$	$14[n-4]_3$
17	6	2	$5[n-3]_3$	$20[n-4]_2$
18	7	4	$2[n-3]_4$	$12[n-4]_3$
19	6	6	$7/3[n-3]_3$	$4[n-4]_2$
20	9	96	$1/48[n-3]_6$	$3/4[n-4]_5$
21	8	16	$1/4[n-3]_5$	$4[n-4]_4$
22	8	48	$1/8[n-3]_5$	$7/6[n-4]_4$
23	6	16	$1/2[n-3]_3$	$3[n-4]_2$
24	7	48	$1/4[n-3]_4$	$2/3[n-4]_3$
25	6	120	$1/6[n-3]_3$	0
26	10	3840	0	$1/48[n-4]_6$

Table 4

$6\backslash 2$			1	2
	$\lvert S\rvert$		3	4
		$\lvert A_S\rvert$	24	8
1	4	24	$1[n-3]$	1
2	6	8	$2[n-3]_3$	$7[n-4]_2$
3	5	2	$10[n-3]_2$	$20[n-4]$
4	5	2	$11[n-3]_2$	$16[n-4]$
5	7	4	$3[n-3]_4$	$18[n-4]_3$
6	6	2	$7[n-3]_3$	$32[n-4]_2$
7	6	1	$16[n-3]_3$	$56[n-4]_2$
8	6	2	$9[n-3]_3$	$24[n-4]_2$
9	8	16	$5/8[n-3]_5$	$5[n-4]_4$
10	7	4	$3[n-3]_4$	$18[n-4]_3$
11	7	4	$7/2[n-3]_4$	$16[n-4]_3$
12	7	8	$2[n-3]_4$	$7[n-4]_3$
13	5	2	$9[n-3]_2$	$24[n-4]$
14	5	8	$5/2[n-3]_2$	$5[n-4]$
15	6	4	$4[n-3]_3$	$14[n-4]_2$
16	6	2	$10[n-3]_3$	$20[n-4]_2$
17	6	6	$3[n-3]_3$	$8[n-4]_2$
18	6	12	$2[n-3]_3$	$2[n-4]_2$
19	7	36	$1/3[n-3]_4$	$2[n-4]_3$
20	6	72	$1/6[n-3]_3$	$1[n-4]_2$
21	7	12	$5/6[n-3]_4$	$20/3[n-4]_3$
22	8	24	$1/3[n-3]_5$	$11/3[n-4]_4$
23	9	288	$1/48[n-3]_6$	$1/3[n-4]_5$
24	8	48	$7/24[n-3]_5$	$4/3[n-4]_4$
25	7	6	$8/3[n-3]_4$	$28/3[n-4]_3$
26	7	12	$3/2[n-3]_4$	$4[n-4]_3$
27	8	4	$5/2[n-3]_5$	$20[n-4]_4$
28	7	2	$6[n-3]_4$	$36[n-4]_3$
29	7	8	$7/4[n-3]_4$	$8[n-4]_3$
30	9	24	$1/3[n-3]_6$	$11/3[n-4]_5$
31	8	12	$5/6[n-3]_5$	$20/3[n-4]_4$
32	8	72	$1/6[n-3]_5$	$1[n-4]_4$
33	6	4	$9/2[n-3]_3$	$12[n-4]_2$
34	7	1	$12[n-3]_4$	$72[n-4]_3$
35	6	2	$7[n-3]_3$	$32[n-4]_2$
36	6	4	$4[n-3]_3$	$14[n-4]_2$
37	8	4	$5/2[n-3]_5$	$20[n-4]_4$
38	7	2	$7[n-3]_4$	$32[n-4]_3$
39	6	2	$8[n-3]_3$	$28[n-4]_2$
40	8	16	$3/4[n-3]_5$	$9/2[n-4]_4$
41	7	16	$5/8[n-3]_4$	$5[n-4]_3$
42	8	4	$2[n-3]_5$	$22[n-4]_4$
43	7	2	$5[n-3]_4$	$40[n-4]_3$
44	9	8	$3/4[n-3]_6$	$12[n-4]_5$
45	8	8	$1[n-3]_5$	$11[n-4]_4$
46	6	2	$7[n-3]_3$	$32[n-4]_2$
47	6	12	$1[n-3]_3$	$6[n-4]_2$
48	8	4	$2[n-3]_5$	$22[n-4]_4$
49	9	48	$1/8[n-3]_6$	$2[n-4]_5$
50	10	64	$1/16[n-3]_7$	$13/8[n-4]_6$
51	5	12	$3/2[n-3]_2$	$4[n-4]$
52	7	4	$3[n-3]_4$	$18[n-4]_3$
53	7	20	$1/2[n-3]_4$	$4[n-4]_3$
54	7	6	$2[n-3]_4$	$12[n-4]_3$
55	9	16	$3/8[n-3]_6$	$6[n-4]_5$
56	8	4	$5/2[n-3]_5$	$20[n-4]_4$
57	7	4	$4[n-3]_4$	$14[n-4]_3$
58	9	16	$1/2[n-3]_6$	$11/2[n-4]_5$
59	8	12	$7/6[n-3]_5$	$16/3[n-4]_4$
60	7	24	$11/12[n-3]_4$	$4/3[n-4]_3$
61	11	768	$1/384[n-3]_8$	$7/48[n-4]_7$
62	10	96	$1/24[n-3]_7$	$13/12[n-4]_6$
63	10	288	$1/48[n-3]_7$	$1/3[n-4]_6$
64	8	64	$1/8[n-3]_5$	$11/8[n-4]_4$
65	9	192	$1/16[n-3]_6$	$3/8[n-4]_5$
66	8	240	$1/12[n-3]_5$	$1/6[n-4]_4$
67	7	720	$1/24[n-3]_4$	0
68	12	46080	0	$1/384[n-4]_8$

Table 5

5\3			1	2	3	4	5
	$\vert S\vert$		3	5	4	4	6
		$\vert A_S\vert$	4	4	2	2	4
1	4	4	$3[n-3]$	0	3	3	0
2	6	4	$3/2[n-3]_3$	$5[n-5]$	$1[n-4]_2$	$3/2[n-4]_2$	12
3	5	2	$3[n-3]_2$	8	$4[n-4]$	$3[n-4]$	0
4	5	2	$3[n-3]_2$	4	$5[n-4]$	$6[n-4]$	0
5	5	4	$3/2[n-3]_2$	1	$2[n-4]$	$6[n-4]$	0
6	6	12	$1/2[n-3]_3$	$3[n-5]$	0	0	0
7	7	48	$1/8[n-3]_4$	$1/2[n-5]_2$	0	0	$3[n-6]$
8	7	12	0	$3[n-5]_2$	0	$1/2[n-4]_3$	0
9	6	2	0	$12[n-5]$	$3[n-4]_2$	$3[n-4]_2$	0
10	6	8	0	$2[n-5]$	$1[n-4]_2$	$3/2[n-4]_2$	0
11	7	4	0	$7[n-5]_2$	$1/2[n-4]_3$	0	$24[n-6]$
12	6	2	0	$12[n-5]$	$3[n-4]_2$	0	24
13	8	16	0	$3/2[n-5]_3$	0	0	$12[n-6]_2$
14	5	2	0	6	$6[n-4]$	$3[n-4]$	0
15	5	10	0	2	$1[n-4]$	0	0
16	7	4	0	$5[n-5]_2$	$1[n-4]_3$	0	$36[n-6]$
17	6	2	0	$8[n-5]$	$4[n-4]_2$	$3[n-4]_2$	24
18	7	4	0	$5[n-5]_2$	$1[n-4]_3$	$3/2[n-4]_3$	$24[n-6]$
19	6	6	0	$2[n-5]$	$1[n-4]_2$	$4[n-4]_2$	0
20	9	96	0	$1/8[n-5]_4$	0	0	$7/2[n-6]_3$
21	8	16	0	$1[n-5]_3$	$1/8[n-4]_4$	0	$15[n-6]_2$
22	8	48	0	$1/2[n-5]_3$	0	$1/8[n-4]_4$	$3[n-6]_2$
23	6	16	0	$1[n-5]$	$1/2[n-4]_2$	0	6
24	7	48	0	$1/2[n-5]_2$	0	$1/2[n-4]_3$	0
25	6	120	0	0	0	$1/2[n-4]_2$	0
26	10	3840	0	0	0	0	$1/8[n-6]_4$

Table 6

6\5			1	2	3	4	5	6	7
	$\vert S\vert$		4	6	5	5	5	6	7
		$\vert A_S\vert$	24	8	2	2	4	2	1
1	4	24	1	0	0	0	0	0	0
2	6	8	$1/2[n-4]_2$	2	0	0	0	0	0
3	5	2	$2[n-4]$	0	2	2	0	0	0
4	5	2	$2[n-4]$	0	0	2	4	0	0
5	7	4	0	$2[n-6]$	0	0	0	$3[n-6]$	0
6	6	2	0	2	$1[n-5]$	0	0	6	0
7	6	1	0	4	$2[n-5]$	$2[n-5]$	0	0	0
8	6	2	0	2	$1[n-5]$	0	$2[n-5]$	0	0
9	8	16	0	$1/2[n-6]_2$	0	0	0	0	$3[n-7]$
10	7	4	0	$1[n-6]$	$1/2[n-5]_2$	0	0	0	12
11	7	4	0	$2[n-6]$	0	$1/2[n-5]_2$	0	0	0
12	7	8	0	$1[n-6]$	0	0	$1/2[n-5]_2$	0	0
13	5	2	0	0	2	1	0	0	0
14	5	8	0	0	1	0	1	0	0
15	6	4	0	0	$1[n-5]$	0	0	3	0
16	6	2	0	0	0	$2[n-5]$	$2[n-5]$	0	0
17	6	6	0	0	0	$1[n-5]$	0	0	0
18	6	12	0	0	0	0	$1[n-5]$	0	0
19	7	36	0	0	0	0	0	$1[n-6]$	0
20	6	72	0	0	0	0	0	1	0
21	7	12	0	0	0	0	0	$2[n-6]$	4
22	8	24	0	0	0	0	0	$1/2[n-6]_2$	$4[n-7]$
23	9	288	0	0	0	0	0	0	$1/2[n-7]_2$

Table 7

$6\backslash 3$			1	2	3	4	5
	$\lvert S\rvert$		3	5	4	4	6
		$\lvert A_S\rvert$	24	8	2	2	4
1	4	24	$1[n-3]$	0	1	1	0
2	6	8	$3/2[n-3]_3$	$4[n-5]$	$3/2[n-4]_2$	$3/2[n-4]_2$	12
3	5	2	$6[n-3]_2$	10	$10[n-4]$	$9[n-4]$	0
4	5	2	$6[n-3]_2$	6	$10[n-4]$	$15[n-4]$	0
5	7	4	$3/2[n-3]_4$	$14[n-5]_2$	$1[n-4]_3$	$3/2[n-4]_3$	$24[n-6]$
6	6	2	$3[n-3]_3$	$24[n-5]$	$5[n-4]_2$	$3[n-4]_2$	24
7	6	1	$6[n-3]_3$	$36[n-5]$	$14[n-4]_2$	$12[n-4]_2$	48
8	6	2	$3[n-3]_3$	$14[n-5]$	$7[n-4]_2$	$12[n-4]_2$	24
9	8	16	$3/8[n-3]_5$	$5/2[n-5]_3$	$1/4[n-4]_4$	$3/8[n-4]_4$	$18[n-6]_2$
10	7	4	$3/2[n-3]_4$	$10[n-5]_2$	$2[n-4]_3$	$3/2[n-4]_3$	$48[n-6]$
11	7	4	$3/2[n-3]_4$	$9[n-5]_2$	$5/2[n-4]_3$	$3[n-4]_3$	$36[n-6]$
12	7	8	$3/4[n-3]_4$	$9/2[n-5]_2$	$1[n-4]_3$	$3[n-4]_3$	$12[n-6]$
13	5	2	$3[n-3]_2$	14	$10[n-4]$	$6[n-4]$	0
14	5	8	$3/2[n-3]_2$	3	$2[n-4]$	$3[n-4]$	0
15	6	4	$3/2[n-3]_3$	$11[n-5]$	$3[n-4]_2$	$3[n-4]_2$	0
16	6	2	$3[n-3]_3$	$12[n-5]$	$8[n-4]_2$	$15[n-4]_2$	0
17	6	6	$1[n-3]_3$	$4[n-5]$	$3[n-4]_2$	$3[n-4]_2$	8
18	6	12	$1/2[n-3]_3$	$1[n-5]$	$1[n-4]_2$	$5[n-4]_2$	0
19	7	36	$1/6[n-3]_4$	$2[n-5]_2$	0	$1/6[n-4]_3$	0
20	6	72	$1/6[n-3]_3$	$1[n-5]$	0	0	0
21	7	12	$1/2[n-3]_4$	$5[n-5]_2$	$1/6[n-4]_3$	0	$12[n-6]$
22	8	24	$1/4[n-3]_5$	$13/6[n-5]_3$	0	0	$12[n-6]_2$
23	9	288	$1/48[n-3]_6$	$1/8[n-5]_4$	0	0	$5/3[n-6]_3$
24	8	48	0	$4/3[n-5]_3$	0	$1/2[n-4]_4$	0
25	7	6	0	$8[n-5]_2$	$4/3[n-4]_3$	$4[n-4]_3$	0
26	7	12	0	$3[n-5]_2$	$1[n-4]_3$	$5/2[n-4]_3$	0
27	8	4	0	$13[n-5]_3$	$1[n-4]_4$	$3/2[n-4]_4$	$48[n-6]_2$
28	7	2	0	$26[n-5]_2$	$4[n-4]_3$	$3[n-4]_3$	$48[n-6]$
29	7	8	0	$7[n-5]_2$	$1[n-4]_3$	$3/2[n-4]_3$	0
30	9	24	0	$13/6[n-5]_4$	0	$1/4[n-4]_5$	$12[n-6]_3$
31	8	12	0	$5[n-5]_3$	$1/6[n-4]_4$	$1/2[n-4]_4$	$12[n-6]_2$
32	8	72	0	$1[n-5]_3$	0	$1/6[n-4]_4$	0
33	6	4	0	$8[n-5]$	$4[n-4]_2$	$6[n-4]_2$	0
34	7	1	0	$44[n-5]_2$	$10[n-4]_3$	$6[n-4]_3$	$144[n-6]$
35	6	2	0	$22[n-5]$	$7[n-4]_2$	$3[n-4]_2$	24
36	6	4	0	$10[n-5]$	$4[n-4]_2$	$3[n-4]_2$	0
37	8	4	0	$11[n-5]_3$	$3/2[n-4]_4$	$3/2[n-4]_4$	$60[n-6]_2$
38	7	2	0	$20[n-5]_2$	$6[n-4]_3$	$6[n-4]_3$	$48[n-6]$
39	6	2	0	$16[n-5]$	$9[n-4]_2$	$6[n-4]_2$	24
40	8	16	0	$5/2[n-5]_3$	$1/2[n-4]_4$	$3/4[n-4]_4$	$12[n-6]_2$
41	7	16	0	$3[n-5]_2$	$1/2[n-4]_3$	0	$12[n-6]$
42	8	4	0	$12[n-5]_3$	$1[n-4]_4$	0	$72[n-6]_2$
43	7	2	0	$24[n-5]_2$	$4[n-4]_3$	0	$96[n-6]$
44	9	8	0	$5[n-5]_4$	$1/4[n-4]_5$	0	$54[n-6]_3$
45	8	8	0	$6[n-5]_3$	$1/2[n-4]_4$	0	$36[n-6]_2$
46	6	2	0	$18[n-5]$	$8[n-4]_2$	$3[n-4]_2$	48
47	6	12	0	$4[n-5]$	$1[n-4]_2$	0	8
48	8	4	0	$10[n-5]_3$	$3/2[n-4]_4$	0	$84[n-6]_2$
49	9	48	0	$1[n-5]_4$	0	0	$8[n-6]_3$
50	10	64	0	$1/2[n-5]_5$	0	0	$9[n-6]_4$
51	5	12	0	2	$2[n-4]$	$1[n-4]$	0
52	7	4	0	$9[n-5]_2$	$3[n-4]_3$	$3/2[n-4]_3$	$48[n-6]$
53	7	20	0	$2[n-5]_2$	$1/2[n-4]_3$	0	$12[n-6]$
54	7	6	0	$6[n-5]_2$	$2[n-4]_3$	$1[n-4]_3$	$32[n-6]$
55	9	16	0	$2[n-5]_4$	$1/4[n-4]_5$	0	$30[n-6]_3$
56	8	4	0	$9[n-5]_3$	$2[n-4]_4$	$3/2[n-4]_4$	$72[n-6]_2$
57	7	4	0	$8[n-5]_2$	$3[n-4]_3$	$6[n-4]_3$	$24[n-6]$
58	9	16	0	$9/4[n-5]_4$	$1/4[n-4]_5$	$3/8[n-4]_5$	$24[n-6]_3$
59	8	12	0	$10/3[n-5]_3$	$1/2[n-4]_4$	$2[n-4]_4$	$12[n-6]_2$
60	7	24	0	$1[n-5]_2$	$1/3[n-4]_3$	$5/2[n-4]_3$	0
61	11	768	0	$1/48[n-5]_6$	0	0	$1[n-6]_5$
62	10	96	0	$1/4[n-5]_5$	$1/48[n-4]_6$	0	$13/2[n-6]_4$
63	10	288	0	$1/8[n-5]_5$	0	$1/48[n-4]_6$	$5/3[n-6]_4$
64	8	64	0	$1/2[n-5]_3$	$1/8[n-4]_4$	0	$6[n-6]_2$
65	9	192	0	$1/4[n-5]_4$	0	$1/8[n-4]_5$	$1[n-6]_3$
66	8	240	0	$1/6[n-5]_3$	0	$1/4[n-4]_4$	0
67	7	720	0	0	0	$1/6[n-4]_3$	0
68	12	46080	0	0	0	0	$1/48[n-6]_6$

Table 8a

$6\backslash4$			1	2	3	4	5	6
	$\lvert S\rvert$		4	5	6	5	5	7
		$\lvert A_S\rvert$	24	8	2	2	4	2
1	4	24	1	0	0	0	0	0
2	6	8	$1[n-4]_2$	$3[n-5]$	0	0	0	0
3	5	2	$5[n-4]$	6	0	4	4	0
4	5	2	$6[n-4]$	0	0	2	5	0
5	7	4	$1/2[n-4]_3$	$6[n-5]_2$	$10[n-6]$	0	0	4
6	6	2	$1[n-4]_2$	$12[n-5]$	12	$4[n-5]$	$1[n-5]$	0
7	6	1	$4[n-4]_2$	$12[n-5]$	8	$8[n-5]$	$6[n-5]$	0
8	6	2	$2[n-4]_2$	$6[n-5]$	0	$2[n-5]$	$5[n-5]$	0
9	8	16	$1/8[n-4]_4$	$3/2[n-5]_3$	0	0	0	$5[n-7]$
10	7	4	$1/2[n-4]_3$	$6[n-5]_2$	0	$1[n-5]_2$	$1/2[n-5]_2$	16
11	7	4	$1[n-4]_3$	$3[n-5]_2$	0	$1/2[n-5]_2$	$1[n-5]_2$	8
12	7	8	$1/2[n-4]_3$	$3/2[n-5]_2$	0	0	$1/2[n-5]_2$	2
13	5	2	$2[n-4]$	6	0	7	4	0
14	5	8	$1[n-4]$	3	0	1	1	0
15	6	4	$1/2[n-4]_2$	$6[n-5]$	6	$2[n-5]$	$2[n-5]$	0
16	6	2	$3[n-4]_2$	0	4	$2[n-5]$	$7[n-5]$	0
17	6	6	$1[n-4]_2$	0	0	$1[n-5]$	$2[n-5]$	0
18	6	12	$1/2[n-4]_2$	0	0	0	$1[n-5]$	0
19	7	36	0	$1[n-5]_2$	$2[n-6]$	0	0	0
20	6	72	0	$1[n-5]$	1	0	0	0
21	7	12	0	$3[n-5]_2$	$4[n-6]$	0	0	4
22	8	24	0	$3/2[n-5]_3$	$1[n-6]_2$	0	0	$6[n-7]$
23	9	288	0	$1/8[n-5]_4$	0	0	0	$1/2[n-7]_2$
24	8	48	0	0	$1[n-6]_2$	0	0	0
25	7	6	0	0	$4[n-6]$	$1[n-5]_2$	$1[n-5]_2$	0
26	7	12	0	0	$2[n-6]$	0	$3/2[n-5]_2$	0
27	8	4	0	0	$8[n-6]_2$	0	$1/2[n-5]_3$	$16[n-7]$
28	7	2	0	0	$16[n-6]$	$3[n-5]_2$	$1[n-5]_2$	8
29	7	8	0	0	$5[n-6]$	$1[n-5]_2$	$1/2[n-5]_2$	0
30	9	24	0	0	$1[n-6]_3$	0	0	$6[n-7]_2$
31	8	12	0	0	$4[n-6]_2$	0	0	$4[n-7]$
32	8	72	0	0	$1[n-6]_2$	0	0	0
33	6	4	0	0	2	$2[n-5]$	$3[n-5]$	0
34	7	1	0	0	$16[n-6]$	$6[n-5]_2$	$4[n-5]_2$	32
35	6	2	0	0	8	$7[n-5]$	$2[n-5]$	0
36	6	4	0	0	4	$3[n-5]$	$2[n-5]$	0
37	8	4	0	0	$2[n-6]_2$	$1[n-5]_3$	$1/2[n-5]_3$	$24[n-7]$
38	7	2	0	0	$8[n-6]$	$2[n-5]_2$	$5[n-5]_2$	8
39	6	2	0	0	4	$4[n-5]$	$6[n-5]$	0
40	8	16	0	0	$1/2[n-6]_2$	0	$1/2[n-5]_3$	$4[n-7]$
41	7	16	0	0	$2[n-6]$	0	0	2
42	8	4	0	0	$6[n-6]_2$	$1/2[n-5]_3$	0	$28[n-7]$
43	7	2	0	0	$12[n-6]$	$3[n-5]_2$	0	24
44	9	8	0	0	$2[n-6]_3$	0	0	$16[n-7]_2$
45	8	8	0	0	$4[n-6]_2$	0	0	$8[n-7]$
46	6	2	0	0	4	$4[n-5]$	$3[n-5]$	0
47	6	12	0	0	2	$1[n-5]$	0	0
48	8	4	0	0	$2[n-6]_2$	$1[n-5]_3$	0	$24[n-7]$
49	9	48	0	0	$1/2[n-6]_3$	0	0	$4[n-7]_2$
50	10	64	0	0	$1/8[n-6]_4$	0	0	$5/2[n-7]_3$
51	5	12	0	0	0	1	1	0
52	7	4	0	0	0	$1[n-5]_2$	$1[n-5]_2$	12
53	7	20	0	0	0	$1/2[n-5]_2$	0	4
54	7	6	0	0	0	$1[n-5]_2$	$1[n-5]_2$	8
55	9	16	0	0	0	$1/8[n-5]_4$	0	$7[n-7]_2$
56	8	4	0	0	0	$1/2[n-5]_3$	$1[n-5]_3$	$16[n-7]$
57	7	4	0	0	0	$1/2[n-5]_2$	$3[n-5]_2$	4
58	9	16	0	0	0	0	$1/8[n-5]_4$	$6[n-7]_2$
59	8	12	0	0	0	0	$1/2[n-5]_3$	$4[n-7]$
60	7	24	0	0	0	0	$1/2[n-5]_2$	0
61	11	768	0	0	0	0	0	$1/8[n-7]_4$
62	10	96	0	0	0	0	0	$1[n-7]_3$
63	10	288	0	0	0	0	0	$1/2[n-7]_3$
64	8	64	0	0	0	0	0	$1[n-7]$
65	9	192	0	0	0	0	0	$1/2[n-7]_2$
66	8	240	0	0	0	0	0	0
67	7	720	0	0	0	0	0	0
68	12	46080	0	0	0	0	0	0

Table 8b

$6\backslash 4$			7	8	9	10	11
	$\lvert S\rvert$		6	6	4	5	8
		$\lvert A_S\rvert$	1	2	16	4	4
1	4	24	0	0	1	0	0
2	6	8	3	3	$1[n-4]_2$	0	0
3	5	2	0	0	$4[n-4]$	0	0
4	5	2	0	0	$4[n-4]$	12	0
5	7	4	$4[n-6]$	$6[n-6]$	0	0	0
6	6	2	6	6	0	0	0
7	6	1	12	12	0	0	0
8	6	2	6	6	0	$12[n-5]$	0
9	8	16	$1[n-6]_2$	$3/2[n-6]_2$	0	0	24
10	7	4	$4[n-6]$	$3[n-6]$	0	0	0
11	7	4	$5[n-6]$	$6[n-6]$	0	0	0
12	7	8	$2[n-6]$	$6[n-6]$	0	$3[n-5]_2$	0
13	5	2	0	0	$4[n-4]$	0	0
14	5	8	0	0	0	3	0
15	6	4	0	3	0	0	0
16	6	2	0	6	0	$12[n-5]$	0
17	6	6	2	0	0	0	0
18	6	12	0	1	0	$10[n-5]$	0
19	7	36	0	$1[n-6]$	0	0	0
20	6	72	0	0	0	0	0
21	7	12	$1[n-6]$	0	0	0	0
22	8	24	0	0	0	0	0
23	9	288	0	0	0	0	$4[n-8]$
24	8	48	0	$2[n-6]_2$	0	$1/2[n-5]_3$	0
25	7	6	0	$10[n-6]$	0	$4[n-5]_2$	0
26	7	12	0	$2[n-6]$	0	$2[n-5]_2$	0
27	8	4	$4[n-6]_2$	$6[n-6]_2$	0	0	0
28	7	2	$8[n-6]$	$12[n-6]$	0	0	0
29	7	8	0	$6[n-6]$	0	0	0
30	9	24	0	$3/2[n-6]_3$	0	0	0
31	8	12	$1[n-6]_2$	$3[n-6]_2$	0	0	0
32	8	72	0	$1[n-6]_2$	0	0	0
33	6	4	0	6	$2[n-4]_2$	$6[n-5]$	0
34	7	1	$20[n-6]$	$12[n-6]$	0	0	0
35	6	2	6	6	0	0	0
36	6	4	0	6	$2[n-4]_2$	0	0
37	8	4	$3[n-6]_2$	$6[n-6]_2$	0	0	0
38	7	2	$8[n-6]$	$6[n-6]$	0	0	0
39	6	2	6	0	$4[n-4]_2$	0	0
40	8	16	$1[n-6]_2$	$3/2[n-6]_2$	0	0	0
41	7	16	$2[n-6]$	0	$1/2[n-4]_3$	0	0
42	8	4	$4[n-6]_2$	0	0	0	0
43	7	2	$12[n-6]$	0	0	0	0
44	9	8	$3/2[n-6]_3$	0	0	0	$96[n-8]$
45	8	8	$3[n-6]_2$	0	0	0	48
46	6	2	12	0	$4[n-4]_2$	0	0
47	6	12	2	0	0	0	0
48	8	4	$5[n-6]_2$	0	0	0	96
49	9	48	0	0	0	0	0
50	10	64	0	0	0	0	$24[n-8]_2$
51	5	12	0	0	$2[n-4]$	0	0
52	7	4	$6[n-6]$	$3[n-6]$	$2[n-4]_3$	0	0
53	7	20	$1[n-6]$	0	0	0	0
54	7	6	$4[n-6]$	0	0	0	0
55	9	16	$1[n-6]_3$	0	0	0	$72[n-8]$
56	8	4	$6[n-6]_2$	$3[n-6]_2$	0	0	96
57	7	4	$4[n-6]$	$6[n-6]$	0	$6[n-5]_2$	0
58	9	16	$1[n-6]_3$	$3/2[n-6]_3$	0	0	$48[n-8]$
59	8	12	$1[n-6]_2$	$5[n-6]_2$	0	$2[n-5]_3$	0
60	7	24	0	$2[n-6]$	0	$5[n-5]_2$	0
61	11	768	0	0	0	0	$9/2[n-8]_3$
62	10	96	$1/8[n-6]_4$	0	0	0	$24[n-8]_2$
63	10	288	0	$1/8[n-6]_4$	0	0	$4[n-8]_2$
64	8	64	$1/2[n-6]_2$	0	$1/8[n-4]_4$	0	12
65	9	192	0	$1/2[n-6]_3$	0	$1/8[n-5]_4$	0
66	8	240	0	$1/2[n-6]_2$	0	$1/2[n-5]_3$	0
67	7	720	0	0	0	$1/2[n-5]_2$	0
68	12	46080	0	0	0	0	$1/8[n-8]_4$

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