Many Isomorphism Types of 6- and 7-Designs

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A simple t-(v, k, λ) design consists of a point set V of size v and a set of blocks $\mathcal{B}$ where each block is a subset of k points and each subset of t points lies in exactly λ blocks.

Until the discovery of 6-$(33, 8, 36)$ designs by Magliveras and Leavitt in the early 80'th [15], it was suspected that simple t-designs might exist only for $t \leq 5$. The big surprise then was the famous Theorem by Teirlinck in 1987 [17] showing that simple t-designs exist for all t. Teirlinck's method is recursive, called large set recursion, and produces point sets of astronomical size.

Since then, successful efforts have been made to directly construct t-designs with "large" t for small point sets and to refine the recursive methods to also reduce the size of the point sets, notably by Ajoodani-Namini [1]. Directly constructed large sets of t-designs are still needed as starting points for recursions. Several t-designs have been constructed by A. Betten, A. Wassermann and the author with their software package DISCRETA for $6 \leq t \leq 9$[3, 4, 2, 6, 13] and further ones are presented here.

We solve the isomorphism problem for directly constructed t-designs with the following theorem.

Theorem 1[16, 8, 13] *Let $G = S_V$ be the full symmetric group on the point set V. Let $A \leq G$ be a group of automorphisms of each t-(v, k, λ) design in a set Ω.*

a) If A is the full automorphism group of each of the designs in Ω then two of these designs are isomorphic only if some permutation in the normalizer

$N_G(A)$ of A maps one design onto the other. If Ω then consists of all t-(v,k,λ) designs with full automorphism group A then the number of different isomorphism types in Ω is $\frac{|\Omega|}{|N_G(A)|/|A|}$.

b) If A contains a Sylow subgroup P of the full automorphism group of each of the designs in Ω then two of these designs are isomorphic only if some permutation in $N_G(P)$ maps one design onto the other.

van Leijenhorst [14] and Tran van Trung [18] construct a new t-design (TvT-construction) from any pair of t-designs whose parameter sets are those of a derived and a residual design from the same parameter set. One can replace one of the two designs by isomorphic copies to obtain further designs with the same parameters. Using double cosets in the full symmetric group on the point set we obtain bounds on the number of isomorphism types from this construction, see [13].

Theorem 2 *If there are n_1 isomorphism types of t-$(v,k-1,\lambda)$ designs and n_2 isomorphism types of t-$(v,k,\lambda(v-k+1)/(k-t))$ designs with automorphism groups of order at most a then there exist at least*

$$\frac{n_1 \cdot n_2 \cdot v!}{(v+1)a^2}$$

isomorphism types of t-$(v+1,k,\lambda(v-t+1)/(k-t))$ designs with automorphism group of order at most $a(v+1)$.

We have constructed new 6-designs and 7-designs and applied Theorems 1 and 2, see the table which also reports the known 6-designs up to 20 points. A $G+$ denotes a group with an added fixed point. A $G-$ denotes a point stabilizer. It should be noted that up to now no 6-design has been found where λ is less than 4. The only 6-designs on less than 19 points known are those listed on 14 points. A 6-$(19,7,5)$ design and any 6-$(19,8,\lambda)$ design are still missing. The TvT-construction is applied iteratively to the 7-designs on 24 points, the previously known 7-$(24,8,6)$, 7-$(24,9,48)$, and 7-$(25,9,54)$ designs [13], and their complementary designs.

The latter parameter sets belong to a *family* of parameter sets that are all obtainable from a unique formally admissible parameter set by computing the parameters of derived designs, residual designs, and regarding a t-design as a s-design for some $s \leq t$. This source of parameter sets itself is neither residual nor derived nor a t-design for some larger t and is called the *ancestor* of the family.

Table of 6-Designs and some 7-Designs

Parameters	*Origin*	*No of Isomorphism Types*	*Reference*
$6-(14,7,4)$	$C_{13}+$	2	[11]
	A_4	1	[12]
	C_3	4	[9]
	C_7	2	[9]
	C_2	4	[9]
$6-(19,7,4)$	$Hol(C_{17})++$	1	[3]
$6-(19,7,6)$	$Hol(C_{19})$	3	[3]
$6-(19,9,126)$	$(\mathbf{PGL(2,8)}\times\mathbf{C_2})+$	4	
$6-(19,9,134)$	$(\mathbf{PSL(2,9)}\times\mathbf{C_2})-$	1	
$6-(20,9,112)$	$PGL(2,19)$	2	[10]
	$\mathbf{PGL(2,9)}\times\mathbf{C_2}$	98	
	$\mathbf{PSL(3,4)}-$	19892	
$6-(20,10,336)$	$\mathbf{PSL(3,4)}-$	$\geq$ 683616	
$6-(20,10,406)$	$7-(20,10,116)$		[3]
	$PSL(2,19)$	7	
	TvT construction with complementary design from 6-(19,9,116)	$\geq$ 13199346316	
$6-(20,10,434)$	$7-(20,10,124)$		[3]
	$PSL(2,19)$	1	[3]
	TvT construction with complementary design from 6-(19,9,124)	$\geq$ 13199346316	
$6-(20,10,441)$	$7-(20,10,126)$		
	$\mathbf{PGL(2,8)}\times\mathbf{C_2}++$	4	
	TvT construction with complementary design from 6-(19,9,126)	$\geq$ 13199346316	
$6-(20,10,469)$	$7-(20,10,134)$		
	$PSL(2,19)$	56	[3]
	TvT construction with complementary design from 6-(19,9,134)	$\geq$ 13199346316	
$6-(20,10,476)$	$\mathbf{PSL(3,4)}-$	$\geq$ 20000000	
$7-(20,10,126)$	$\mathbf{PGL(2,8)}\times\mathbf{C_2}++$	4	
$7-(20,10,134)$	$\mathbf{PGL(2,9)}\times\mathbf{C_2}$	1	
$7-(24,11,840)$	$\mathbf{PGL(2,23)}$	$\geq$ 155	
$7-(24,12,2184)$	$\mathbf{PGL(2,23)}$	$\geq$ 28500	

Parameter Sets of 7-Designs in the Family of 15-$(32,16,6)$

7-(32,16,721050) >= 1.5E4165

7-(31,15,259578) >= 4.0E2080
7-(31,16,461472) >= 4.0E2080

7-(30,14,86526) >= 9.7E1035
7-(30,15,173052) >= 1.3E1039
7-(30,16,288420) >= 9.7E1035

7-(29,13,26334) >= 3.0E511
7-(29,14,60192) >= 3.8E514
7-(29,15,112860) >= 3.8E514
7-(29,16,175560) >= 3.0E511

7-(28,12,7182) >= 3.9E252
7-(28,13,19152) >= 2.8E250
7-(28,14,41040) >= 5.0E255
7-(28,15,71820) >= 2.8E250
7-(28,16,103740) >= 3.9E252

7-(27,11,1710) >= 1.5E125
7-(27,12,5472) >= 6.2E117
7-(27,13,13680) >= 1.1E123
7-(27,14,27360) >= 1.1E123
7-(27,15,44460) >= 6.2E117
7-(27,16,59280) >= 1.5E125

7-(26,10,342) >= 1E62
7-(26,11,1368) >= 6.6E51
7-(26,12,4104) >= 4.0E54
7-(26,13,9576) >= 1.2E57
7-(26,14,17784) >= 4.0E54
7-(26,15,26676) >= 6.6E51
7-(26,16,32604) >= 1E62

7-(25,9,54) >= 1.2E20
7-(25,10,288) >= 2.3E19
7-(25,11,1080) >= 4.5E19
7-(25,12,3024) >= 1.4E22
7-(25,13,6552) >= 1.4E22
7-(25,14,11232) >= 4.5E19
7-(25,15,15444) >= 2.3E19
7-(25,16,17160) >= 1.2E20

7-(24,8,6) >= 132
7-(24,9,48) >= 2827
7-(24,10,240) >= 91
7-(24,11,840) >= 155
7-(24,12,2184) >= 28500
7-(24,13,4368) >= 155
7-(24,14,6864) >= 91
7-(24,15,8580) >= 2827
7-(24,16,8580) >= 132

Theorem 3 *Each parameter set of a non-trivial design belongs to a finite family with a unique ancestor.*

For all parameter sets of 7-designs in the family of 15-$(32, 16, 6)$ we could establish the existence of such designs. We present these parameter sets in a table where a lower bound for the number of isomorphism types is given below each parameter set. No 8-design is known with parameters from this family. For further families see [5].

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