Some simple 7-designs

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Abstract

Some simple 7-designs with small parameters are constructed with the aid of a computer. The smallest parameter set found is 7-$(24, 8, 4)$. An automorphism group is prescribed for finding the designs and used for determining the isomorphism types. Further designs are derived from these designs by known construction processes.

1 Parameter Sets

Certain projective groups are 3-homogeneous and have a small number of orbits on k-subsets for moderately small k. They have therefore been a valuable tool in several geometric constructions. The first simple 6-designs were found by Magliveras and Leavitt using a prescribed automorphism group $\mathrm{P\Gamma L}(2, 32)$, [13]. Later, further 6-designs were found having other projective automorphism groups, see [7], [15], [9]. For a recent survey on t-designs with large t see D. L. Kreher's contribution in [10]. The recipe used to construct these designs in principle also applies to the construction of simple 7-designs.

Theorem 1.1 *The following projective groups are automorphism groups of t-(v, k, λ) designs*:

I $\mathrm{PSL}(2, 23)$ *of* 7-$(24, 8, \lambda)$, *where* $\lambda = 4,\ 5,\ 6,\ 7,\ 8$;

II $\mathrm{PGL}(2, 23)$ *of* 7-$(24, 9, \lambda)$, *where* $\lambda = 40,\ 48,\ 64$;

III $\mathrm{PGL}(2, 25)$ *of* 7-$(26, 8, 6)$,

IV $\mathrm{P\Gamma L}(2, 25)$ *of* 7-$(26, 9, \lambda)$, $\lambda = 54,\ 63,\ 81$;

V $\mathrm{P\Gamma L}(2, 32)$ *of* 7-$(33, 8, 10)$, [*2*].

The only 7-designs known before were those of Teirlinck [16] with $k = t+1$ and astronomically large λ and v like :

$$\lambda \ = \ (t+1)!^{2t+1}, \ v \equiv t \ mod \ \lambda.$$

Applying a construction from Tran van Trung[17], see also Kreher[9], yields further 7-designs from those of the theorem.

Corollary 1.2 *There exist simple 7-designs with the following parameter sets:*

VI 7-$(25, 9, \lambda)$ *for* $\lambda = 45, 54, 72$;

VII 7-$(27, 9, 60)$.

Thus, there exist simple 7-designs for $v = 24, 25, 26, 27$, and 33 with some projective automorphism groups.

2 Methods

The designs are constructed by the Kramer-Mesner method [6]. This method assumes a prescribed group A of automorphisms of the desired t-(v, k, λ) designs. The group A is a permutation group on the underlying set V of v elements acting in the induced way on the set of all k-subsets of V. A design allows A as an automorphism group if and only if the set of blocks of the design consists of full k-orbits of A.

Therefore a collection of such k-orbits has to be chosen such that each t-subset T is contained in equally many blocks from these orbits. So for each k-orbit K^A the number $m(T, K^A)$ of members containing T is computed. If T is replaced by some T' from the t-orbit T^A these numbers remain unchanged. So it suffices to consider only one representative T from each t-orbit. There results a matrix M with a row for each t-orbit and a column for each k-orbit. Choosing k-orbits for a t-(v, k, λ) design means to multiply M by an appropriate 0/1-vector on the right such that a vector with constant entries λ results.

There have been different approaches to finding such 0/1 vectors. We have implemented a variant of the LLL-algorithm [12], see [18], which in comparison to Kreher and Radzizowski [8] has the new feature of considering λ as a variable. This helps find unsuspected values of λ. After applying the LLL-algorithm all solutions are determined by an exhaustive search. The Kramer-Mesner matrix is computed by a new version of B. Schmalz's Leiterspiel (snakes and ladders). Our computational system *DISCRETA* allows the choice of groups A from some predefined series. The user computes the matrices and solves the diophantine system of equations by pressing some buttons at a graphical user interface. Besides the LLL-solver we have also included in the system a clever backtrack-solver written by B.D. McKay [14] and a linear programming tool lp-solve [1]. McKay's solver, in particular, is frequently a valuable alternative to the LLL method. The system is written in C and uses a Motif package for the graphical surface. It can be obtained from the authors via ftp.

The following group theoretic results allow us to determine the isomorphism types of designs with prescribed automorphism groups in many important cases without isomorphism testing.

Theorem 2.1 *Let G be a group acting on a set Ω. Let A be a subgroup of G which is the full stabilizer of the points in a set $\Delta \subseteq \Omega$. Then two points of Δ may only lie in the same G-orbit if they lie in the same orbit of $N_G(A)$, the normalizer of A in G.*

If Δ in the theorem is the set of all points having stabilizer A then $N_G(A)$ acts on this set with orbits of length $|N_G(A)/A|$. Thus, the number of isomorphism types of designs having a prescribed full automorphism group A is obtained by dividing the total number of all designs having a prescribed full automorphism group A by the index of A in its normalizer taken in the full symmetric group on the underlying point set. If the group A is not the full automorphism group of some designs fixed by A then those designs must have a larger automorphism group. The principle of inclusion-exclusion allows to determine the number of isomorphism types with prescribed automorphism group in this situation. This is the method W. Burnside formalized with his table of marks [3] for general actions of finite groups, see also [15],[11] for constructive aspects of this approach. Since in many situations the subgroups which occur as stabilizers are not easy to determine, the following special situation is of interest.

Theorem 2.2 *Let G be a group acting on a set Ω. Let $\omega_1, \omega_2 \in \Omega$ and let P be a Sylow-p-subgroup of G fixing ω_1 and ω_2. Then if ω_1 and ω_2 are in the same orbit of G both points are already in the same orbit of $N_G(P)$, the normalizer of P in G.*

In the situation of the theorem for a subgroup A containing P no knowledge about the overgroups of A is needed to decide whether two points fixed by A lie in the same G-orbit. The only difficulty in formulating general counting formulas results from the fact that the set of points fixed by A usually is not closed under $N_G(P)$. However, it is sometimes possible to enlarge P and $N_G(P)$ so that the overgroup of $N_G(P)$ acts on the set of fixed points of the overgroup of P. Hence, for any prime p the projective group $\mathrm{PSL}(2,p)$ contains a Sylow-p-subgroup P of S_{p+1} and $\mathrm{PGL}(2,p)$ contains the normalizer of P. Therefore the following holds.

Corollary 2.3 *For any prime p all designs which admit $\mathrm{PGL}(2,p)$ as a group of automorphisms are pairwise non-isomorphic. All designs admitting $\mathrm{PSL}(2,p)$ but not $\mathrm{PGL}(2,p)$ as a group of automorphisms are grouped into isomorphic pairs under the action of $\mathrm{PGL}(2,p)/\mathrm{PSL}(2,p)$.*

By the preceding methods, the numbers of designs obtained for the cases I to V yield the following numbers of isomorphism types:

I 1, 138, ≥ 590, $\geq 126, \geq 65$ for 7-$(24, 8, \lambda)$ and $\lambda = 4, 5, 6, 7, 8$;

II 113, 5463, ≥ 15325 for 7-$(24, 9, \lambda)$, where $\lambda = 40, 48, 64$;

III 7 for 7-$(26, 8, 6)$,

IV 3989, 37932, ≥ 14 for 7-$(26, 9, \lambda)$, where $\lambda = 54, 63, 81$;

V 4996426 for 7-$(33, 8, 10)$, [18].

The corollary also explains why in their investigations of Steiner 5-designs M.J. Grannel, T.S. Griggs and R.A. Mathon in a series of papers, see [4], always found two copies of each isomorphism type of Steiner systems with some prescribed automorphism group PSL$(2, p)$.

B.D. McKay [14] was the first to find more 7-$(33, 8, 10)$ designs different from those in [2]. He estimated the existence of about 5 million designs of type V, and this gave the impetus for the development of better equation solver for the Kramer-Mesner method, see [18]. In fact, there are 4996426 such designs which is suprisingly close to his estimate.

A detailed presentation of all results mentioned would be very space consuming. A moderate listing is planned to appear elsewhere together with some new 6-designs and material on deduced parameter sets. The details can be obtained from the authors, see also our WWW-pages. Here we include only one representative for the smallest value of λ in each of the basic cases I - IV.

3 Selected 7-Designs with Small λ

I: We use the following permutation representation of PGL$(2, 23)$, a group of order 12144. Generators are the permutations

$\alpha = (3\ 7\ 4\ 12\ 6\ 22\ 10\ 19\ 18\ 13\ 11\ 24\ 20\ 23\ 15\ 21\ 5\ 17\ 8\ 9\ 14\ 16\ )$
$\beta = (3\ 16\ 14\ 9\ 8\ 17\ 5\ 21\ 15\ 23\ 20\ 24\ 11\ 13\ 18\ 19\ 10\ 22\ 6\ 12\ 4\ 7)$
$\gamma = (2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24)$
$\delta = (1\ 3\ 14\ 10\ 8\ 16\ 6\ 12\ 5\ 20\ 9\ 23\ 4\ 18\ 7\ 22\ 15\ 21\ 11\ 19\ 17\ 13\ 24)$ The permutations β^2, γ, and δ generate PSL$(2, 23)$, a group of order 6072.

The 7-$(24, 8, 4)$ design consists of the orbits of the following 8-subsets, called *starter blocks*, under the action of PSL$(2, 23)$:

Starter Blocks								Orbit Length
17	18	19	20	21	22	23	24	6072
7	18	19	20	21	22	23	24	6072
5	1	19	20	21	22	23	24	3036
9	1	19	20	21	22	23	24	6072
5	4	19	20	21	22	23	24	6072
16	4	19	20	21	22	23	24	6072
10	8	19	20	21	22	23	24	3036
16	8	19	20	21	22	23	24	6072
10	3	1	20	21	22	23	24	3036
13	3	1	20	21	22	23	24	3036
17	3	1	20	21	22	23	24	6072
11	4	1	20	21	22	23	24	6072
16	4	1	20	21	22	23	24	6072
17	4	1	20	21	22	23	24	3036
12	5	1	20	21	22	23	24	6072
18	5	1	20	21	22	23	24	6072
9	7	1	20	21	22	23	24	6072
13	8	1	20	21	22	23	24	6072
8	5	3	20	21	22	23	24	6072
10	5	3	20	21	22	23	24	6072
8	6	3	20	21	22	23	24	6072
13	6	3	20	21	22	23	24	6072
13	7	3	20	21	22	23	24	6072
17	8	3	20	21	22	23	24	3036
13	10	3	20	21	22	23	24	3036
14	11	3	20	21	22	23	24	3036
16	14	3	20	21	22	23	24	6072
11	6	4	20	21	22	23	24	6072
18	6	4	20	21	22	23	24	6072
13	8	4	20	21	22	23	24	3036
18	13	4	20	21	22	23	24	3036
10	7	5	20	21	22	23	24	3036
11	8	5	20	21	22	23	24	3036
12	9	5	20	21	22	23	24	1518
16	10	5	20	21	22	23	24	3036
16	10	7	20	21	22	23	24	3036
17	14	12	20	21	22	23	24	759
18	14	6	4	21	22	23	24	759

II: One out of 113 isomorphism types of 7-$(24, 9, 40)$ designs has the following starter blocks for orbits under the action of $\mathrm{PGL}(2, 23)$:

Starter Blocks									Orbit Length
16	17	18	19	20	21	22	23	24	6072
6	17	18	19	20	21	22	23	24	12144
8	17	18	19	20	21	22	23	24	12144
9	17	18	19	20	21	22	23	24	6072
8	1	18	19	20	21	22	23	24	12144
7	4	18	19	20	21	22	23	24	12144
8	4	18	19	20	21	22	23	24	12144
9	4	18	19	20	21	22	23	24	12144
12	5	18	19	20	21	22	23	24	4048
13	5	18	19	20	21	22	23	24	6072
9	8	18	19	20	21	22	23	24	12144
5	3	1	19	20	21	22	23	24	12144
12	3	1	19	20	21	22	23	24	6072
17	3	1	19	20	21	22	23	24	2024
9	4	1	19	20	21	22	23	24	12144
16	4	1	19	20	21	22	23	24	6072
11	5	1	19	20	21	22	23	24	12144
12	8	1	19	20	21	22	23	24	2024
10	9	1	19	20	21	22	23	24	12144
8	5	4	19	20	21	22	23	24	12144
10	5	4	19	20	21	22	23	24	12144
11	5	4	19	20	21	22	23	24	2024
16	5	4	19	20	21	22	23	24	12144
10	9	4	19	20	21	22	23	24	12144
16	10	4	19	20	21	22	23	24	6072
7	4	3	1	20	21	22	23	24	12144
13	4	3	1	20	21	22	23	24	12144
8	5	3	1	20	21	22	23	24	4048
11	5	3	1	20	21	22	23	24	12144
16	7	3	1	20	21	22	23	24	12144
13	12	3	1	20	21	22	23	24	12144
7	5	4	1	20	21	22	23	24	6072
9	5	4	1	20	21	22	23	24	12144
13	5	4	1	20	21	22	23	24	12144
9	8	4	1	20	21	22	23	24	12144
13	8	5	1	20	21	22	23	24	12144
14	8	5	1	20	21	22	23	24	12144
16	8	5	1	20	21	22	23	24	6072
13	8	5	3	20	21	22	23	24	6072
15	10	5	3	20	21	22	23	24	12144

III: We use the following permutation representation of $P\Gamma L(2,25)$, a group of order 31200. Generators are the permutations

$$\begin{aligned}
\alpha &= (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24)\\
\beta &= (1\ 17\ 14\ 15\ 10)(2\ 5\ 13\ 22\ 3)(4\ 11\ 9\ 19\ 8)(6\ 18\ 12\ 25\ 24)(7\ 21\ 23\ 16\ 20)\\
\gamma &= (1\ 8\ 4\ 17\ 3)(2\ 21\ 22\ 19\ 11)(5\ 16\ 20\ 13\ 15)(6\ 12\ 26\ 24\ 18)(7\ 10\ 9\ 14\ 23)\\
\delta &= (1\ 5)(2\ 10)(3\ 15)(4\ 20)(7\ 11)(8\ 16)(9\ 21)(13\ 17)(14\ 22)(19\ 23)
\end{aligned}$$

The permutations α, β, and γ generate $PGL(2,25)$, a group of order 15600.

One out of 7 isomorphism types of 7-$(26,8,6)$ designs has the following starter blocks for orbits under the action of $PGL(2,25)$:

Starter Blocks								Orbit Length
2	20	21	22	23	24	25	26	15600
7	20	21	22	23	24	25	26	15600
8	20	21	22	23	24	25	26	15600
3	2	21	22	23	24	25	26	7800
4	2	21	22	23	24	25	26	15600
7	2	21	22	23	24	25	26	15600
9	2	21	22	23	24	25	26	15600
18	2	21	22	23	24	25	26	15600
10	3	21	22	23	24	25	26	15600
11	3	21	22	23	24	25	26	15600
6	4	21	22	23	24	25	26	15600
8	4	21	22	23	24	25	26	7800
9	4	21	22	23	24	25	26	15600
16	4	21	22	23	24	25	26	7800
11	5	21	22	23	24	25	26	15600
15	5	21	22	23	24	25	26	15600
16	5	21	22	23	24	25	26	7800
9	6	21	22	23	24	25	26	7800
14	6	21	22	23	24	25	26	15600
15	6	21	22	23	24	25	26	3900
9	7	21	22	23	24	25	26	15600
10	9	21	22	23	24	25	26	7800
11	10	21	22	23	24	25	26	1950
5	4	2	22	23	24	25	26	7800
10	4	2	22	23	24	25	26	7800
13	4	2	22	23	24	25	26	15600
18	4	2	22	23	24	25	26	15600
11	5	2	22	23	24	25	26	15600
17	5	2	22	23	24	25	26	15600
14	6	2	22	23	24	25	26	15600
16	6	2	22	23	24	25	26	15600
19	6	2	22	23	24	25	26	15600
16	8	2	22	23	24	25	26	7800
18	10	2	22	23	24	25	26	15600
19	10	2	22	23	24	25	26	7800
19	11	2	22	23	24	25	26	7800
5	4	3	22	23	24	25	26	3900
9	7	3	22	23	24	25	26	3900
14	9	3	22	23	24	25	26	7800
17	5	4	22	23	24	25	26	15600
12	8	4	22	23	24	25	26	7800
17	10	4	3	23	24	25	26	3900

IV: One out of 3989 isomorphism types of 7-$(26,9,54)$ designs consisting of the orbits of the following starter blocks under the action of $P\Gamma L(2,25)$:

Starter Blocks									Orbit Length
18	19	20	21	22	23	24	25	26	15600
2	19	20	21	22	23	24	25	26	31200
4	19	20	21	22	23	24	25	26	31200
8	2	20	21	22	23	24	25	26	31200
15	2	20	21	22	23	24	25	26	15600
17	2	20	21	22	23	24	25	26	31200
8	3	20	21	22	23	24	25	26	15600
5	4	20	21	22	23	24	25	26	15600
6	4	20	21	22	23	24	25	26	15600
7	4	20	21	22	23	24	25	26	31200
8	4	20	21	22	23	24	25	26	31200
9	4	20	21	22	23	24	25	26	31200
10	4	20	21	22	23	24	25	26	15600
12	4	20	21	22	23	24	25	26	31200
13	5	20	21	22	23	24	25	26	15600
8	6	20	21	22	23	24	25	26	31200
10	6	20	21	22	23	24	25	26	15600
14	6	20	21	22	23	24	25	26	15600
9	8	20	21	22	23	24	25	26	31200
7	3	2	21	22	23	24	25	26	31200
9	3	2	21	22	23	24	25	26	31200
11	4	2	21	22	23	24	25	26	31200
13	4	2	21	22	23	24	25	26	15600
11	5	2	21	22	23	24	25	26	31200
8	6	2	21	22	23	24	25	26	31200
9	6	2	21	22	23	24	25	26	31200
14	7	2	21	22	23	24	25	26	31200
15	7	2	21	22	23	24	25	26	31200
15	8	2	21	22	23	24	25	26	15600
13	9	2	21	22	23	24	25	26	15600
14	9	2	21	22	23	24	25	26	15600
11	10	2	21	22	23	24	25	26	3900
15	13	2	21	22	23	24	25	26	31200
18	17	2	21	22	23	24	25	26	31200
17	12	3	21	22	23	24	25	26	7800
8	7	4	21	22	23	24	25	26	15600
17	7	4	21	22	23	24	25	26	31200
13	10	4	21	22	23	24	25	26	15600
8	7	5	21	22	23	24	25	26	31200
15	7	6	21	22	23	24	25	26	7800
19	10	6	2	22	23	24	25	26	15600
20	11	6	2	22	23	24	25	26	31200

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