Steiner Systems with Automorphism Groups $PSL(2,71)$, $PSL(2,83)$, and $P\Sigma L(2,3^5)$

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Abstract

The first 5-$(72,6,1)$ designs with automorphism group $PSL(2,71)$ have been found by Mills [10]. We enumerated all 5-$(72,6,1)$ designs with this automorphism group. There are 926299 non-isomorphic designs.

We show that a necessary condition for semiregular 5-$(v,6,1)$ designs with automorphism group $PSL(2,v-1)$ to exist is $v \equiv 84, 228 \pmod{360}$. There are exactly 3 non-isomorphic semiregular 5-$(84,6,1)$ designs with automorphism group $PSL(2,83)$.

There are at least 6450 non-isomorphic 5-$(244,6,1)$ designs with automorphism group $P\Sigma L(2,3^5)$.

1 Introduction

For the construction of t-(v,k,λ) designs the approach of Kramer and Mesner [7] has been very successful: At first a automorphism group G is prescribed and the incidence matrix $A^G_{t,k}$ of the orbits is calculated. Then, a design having G as a group of automorphisms corresponds to solutions x of the Diophantine linear system

$$A^G_{t,k} \cdot x = \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix},$$

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where x is a 0/1-vector. The solving of this system is a NP-complete task. Finally, isomorphic designs have to be identified. The first two steps can be done with DISCRETA [1], a software package developed by the authors. The isomorphism problem is solved by the methods in [3].

Steiner systems with $t > 3$ are still rare objects. It is not known whether any exist for $t \geq 6$, and for $t = 5$ only a few parameter sets are known. All known Steiner 4-systems are derived from Steiner 5-systems. So, we continue the search for such objects.

In the search for Steiner systems with large t, i. e. 5-$(v, k, 1)$ designs, a fruitful approach was to further cut the search space by restricting the incidence matrix $A^G_{t,k}$ of the orbits to orbits of k-subsets which do not have length equal to the group order. These orbits usually are called *short* orbits.

The values of v for which 5-$(v, 6, 1)$ designs are known, are 12, 24, 36, 48, 72, 84, 108, 132, 168. Apart from the recently found 5-$(36, 6, 1)$ design [2] they all admit some $PSL(2, q)$ as a group of automorphisms, where $q \equiv 3 \bmod 4$. Their number of isomorphism types was known only for $v \leq 48$ completely and — restricted to short orbit-designs — also for $v = 72$, 84. Denniston [5] showed that $q \equiv 3 \bmod 4$ is a necessary condition for the existence of t-$(q + 1, t + 1, 1)$ designs with odd $t \geq 5$, having $PSL(2, q)$ as a group of automorphisms, and that $PGL(2, q)$ never occurs as group of automorphisms for t-$(q + 1, t + 1, 1)$ designs.

For 5-$(72, 6, 1)$ designs with automorphism group $PSL(2, 81)$ we could drop the restriction to short orbits and enumerate all non-isomorphic designs having this group as automorphism group.

We also tried the opposite restriction to use only long orbits to reduce the search space. Such Steiner systems then are *semiregular* designs. Since most orbits usually are long orbits, one would expect a large number of solutions. But it is easy to see that already divisibility conditions heavily restrict the possible situations where such designs might exist. We give a necessary condition for the existence of parameter sets of semiregular Steiner 5-$(v, 6, 1)$ designs with automorphism group $PSL(2, q)$ for some prime power q and consider the smallest possible case, i. e. $v = 84$. Surprisingly, there only exist exactly 3 isomorphism types in this case. The next smallest parameter set for a semiregular Steiner 5-(v,6,1) design would be 5-$(228, 6, 1)$.

Since already in the case of the famous Witt designs the full automorphism group of a Steiner 5-design was much bigger than the corresponding $PSL(2, p)$, we also used a bigger group to find 5-$(244, 6, 1)$ designs. A bigger group as a rule reduces the size of of the Diophantine linear system whose solutions are the designs in the number of rows and in the number of columns roughly by the factor of the index in that group.

2 5-$(72, 6, 1)$ DESIGNS

There had been some success in prescribing that only short orbits should be contained in the Steiner systems. So, the number of possibilities was greatly reduced and the full number of isomorphism types with this additional property could be determined. The first 5-$(72, 6, 1)$ designs have been found by Mills [10] using this approach, and up to 8 designs with this parameter set consisting only of short orbits are known since B. Schmalz [11].

Grannell, Griggs and Mathon [6] found that there exist exactly 4204 isomorphism types with blocks from short orbits only.

The present version of DISCRETA now was able to determine the full set of all isomorphism types of 5-$(72, 6, 1)$ designs with automorphism group $PSL(2, 71)$. There exist exactly 926299 isomorphism types. The order of the group $PSL(2, 71)$ is equal to 178920. The incidence matrix of the orbits has 79 rows and 982 columns.

3 5-$(84, 6, 1)$ DESIGNS

Grannell, Griggs and Mathon [6] showed that for short orbits there are exactly 38717 isomorphism types. There will be much more isomorphism types if we take into account orbits of arbitrary length. We already enumerated at least 348512 isomorphism types. So, we look at the other extreme of a restriction, i. e. to use only orbits of full length.

If a design admits a group of automorphisms G then its set of blocks consists of a collection of orbits on k-subsets. The smallest possible number of orbits is achieved with semiregular designs, i. e. if each orbit has the length $|G|$.

Theorem 3.1 *If there exists a 5-(q+1,6,1) design which is semiregular under the automorphism group $PSL(2, q)$, q odd, then $q \equiv 83, 227$ mod 360.*

Proof Assume, a design with these properties exists. Then the number of blocks b must be divisible by the group order. Thus, we obtain that the following fraction represents a natural number.

$$\frac{b}{|PSL(2,q)|} = \frac{\binom{v}{t}/\binom{k}{t}}{(q+1)q(q-1)/2}$$

where $v = q + 1$, $k = 6$, and $t = 5$. Therefore,

$$\frac{(q-2)(q-3)}{5 \cdot 8 \cdot 9}$$

must be a natural number. But since q is a prime power, $q-3$ cannot be divisible by 9. $(q-2)$ and $(q-3)$ are coprime, so 9 has to divide $(q-2)$. Similarly, 8 divides $(q-3)$. Lastly, 5 divides either $(q-2)$ or $(q-3)$. By the Chinese remainder theorem we have a unique solution modulo $5 \cdot 8 \cdot 9 = 360$ in each case. So, 227 and 83 are the unique solutions mod 360, respectively.

□

For the smallest case $v = 84$ we have used DISCRETA [1] to find all 5-(84,6,1) designs which consist only of orbits of length $|PSL(2, 83)|$ and found exactly 6 solutions. These are grouped into 3 isomorphic pairs under the action of $PGL(2, 83)$ such that there exist exactly 3 isomorphism types by [3]. Such a semiregular 5-(84,6,1) design has exactly 18 block orbits. We list representatives of these orbits for each of the designs.

Design 1 and design 3, respectively design 2 and design 3 are pairwise disjoint such that they can be combined to designs with $\lambda = 2$. Since the group $PSL(2, 83)$ acts 3-homogeneously, each orbit of these semiregular designs is a 3-$(84, 6, 60)$ design. This means, each Steiner system can be partitioned into 18 3-designs. The designs can not be partitioned into 4-designs with automorphism group $PSL(2, 83)$.

We used the following generators of $PSL(2, 83)$, its order is equal to 285852:

(1 82)(2 41)(3 55)(4 62)(5 33)(6 69)(7 71)(8 31)(9 46)(10 58)(11 15)(12 76)
(13 51)(14 77)(16 57)(17 39)(18 23)(19 48)(20 29)(21 79)(22 49)(24 38)(25 73)(26 67),

(1 41 81)(2 55 40)(3 62 54)(4 33 61)(5 69 32)(6 71 68)(7 31 70)(8 46 30)(9 58 45)
(10 15 57)(11 76 14)(12 51 75)(13 77 50)(16 39 56)(17 23 38).

The block-orbit representatives:

Design 1:	Design 2:	Design 3:
$\{1,2,3,4,5,38\}$	$\{1,2,3,4,5,51\}$	$\{1,2,3,4,5,15\}$
$\{1,2,3,4,6,20\}$	$\{1,2,3,4,6,9\}$	$\{1,2,3,4,7,77\}$
$\{1,2,3,4,7,11\}$	$\{1,2,3,4,7,12\}$	$\{1,2,3,4,8,40\}$
$\{1,2,3,4,9,62\}$	$\{1,2,3,4,8,20\}$	$\{1,2,3,4,10,54\}$
$\{1,2,3,4,10,46\}$	$\{1,2,3,4,10,72\}$	$\{1,2,3,4,12,68\}$
$\{1,2,3,4,12,44\}$	$\{1,2,3,4,14,65\}$	$\{1,2,3,4,13,72\}$
$\{1,2,3,4,14,29\}$	$\{1,2,3,4,15,43\}$	$\{1,2,3,4,16,48\}$
$\{1,2,3,4,17,47\}$	$\{1,2,3,4,22,63\}$	$\{1,2,3,4,20,34\}$
$\{1,2,3,4,24,42\}$	$\{1,2,3,4,23,53\}$	$\{1,2,3,4,23,76\}$
$\{1,2,3,4,22,27\}$	$\{1,2,3,4,31,62\}$	$\{1,2,3,4,24,66\}$
$\{1,2,3,4,31,63\}$	$\{1,2,3,4,44,76\}$	$\{1,2,3,4,35,84\}$
$\{1,2,3,4,35,65\}$	$\{1,2,3,4,45,54\}$	$\{1,2,3,4,37,63\}$
$\{1,2,3,5,6,13\}$	$\{1,2,3,5,9,52\}$	$\{1,2,3,4,45,62\}$
$\{1,2,3,5,9,52\}$	$\{1,2,3,5,22,25\}$	$\{1,2,3,5,10,22\}$
$\{1,2,3,5,16,21\}$	$\{1,2,3,5,23,70\}$	$\{1,2,3,5,26,55\}$
$\{1,2,3,5,24,59\}$	$\{1,2,3,5,24,57\}$	$\{1,2,3,5,49,60\}$
$\{1,2,3,5,25,32\}$	$\{1,2,3,5,26,63\}$	$\{1,2,3,6,9,34\}$
$\{1,2,3,6,7,42\}$	$\{1,2,3,6,7,27\}$	$\{1,2,3,6,41,70\}$

4 5-$(244, 6, 1)$ designs

There are only finitely many 5-$(v, 6, 1)$ designs known, see [4]. As the 5-$(36, 5, 1)$ design shows, $v - 1$ need not be a prime power. So, the existence of an automorphism group $PSL(2, v - 1)$ cannot be a necessary condition. It is also not sufficient, since no 5-$(28, 6, 1)$ design exists with automorphism group $PSL(2, 3^3)$. We remark that also no 5-$(82, 6, 1)$ design admitting automorphism group $PSL(2, 81)$ exists. The next power of 3 that is congruent to 3 mod 4 is $q = 3^5$. We find that in this case there do exist Steiner 5-designs, they even admit $P\Sigma L(2, 3^5)$ as a group of automorphisms in which $PSL(2, 3^5)$ has index 5. The order of the group is equal to 35871660, the matrix then still has 196 rows and 7940 columns. So we restricted the search as usual to short orbits only and ended up with 504 columns. This led to at least 12900 solutions.

From [8] we conclude that the only overgroups of $P\Sigma L(2,3^5)$ in S_{3^5+1} are A_{3^5+1} and $P\Gamma L(2,3^5)$. Both groups are not admitted as an automorphism group of a 5-(244, 6, 1) design. Therefore, by [3, Theorem 2.1], the isomorphism types are determined by the action of $P\Gamma L(2,3^5)$ on the set of the designs. So, they fall into orbits of size 2 under this group. The solutions found represent at least 6450 isomorphism types.

It seems interesting to notice that this is the first case of a Steiner 5-$(v,6,1)$ design with an automorphism group $P\Sigma L(2,p^f)$, where $f > 1$, and the first parameter set of a 5-$(v,6,1)$ design where v is not a multiple of 12. Further, this seems to be the first known Steiner 5-designs defined on more than 200 points. So, the number of points is the largest of all presently known Steiner 5-designs.

We used the following generators of the group $P\Sigma L(2,3^5)$:

(3 219 243 82 197 106 40 113 177 155 19 193 133 43 33 142 44 86 174 238 135 70 36 59 171 74 10
192 240 162 73 199 213 159 153 72 63 62 91 201 241 55 194 186 156 233 107 93 229 134 96 146 18
57 225 80 90 65 9 56 8 85 121 122 178 48 221 189 76 117 68 172 210 239 188 102 227 27 58 118 204
158 179 101 13 110 14 166 130 123 149 181 211 132 150 152 98 92 12 220 136 203 187 49 114 151 45 60
145 207 78 144 71 89 94 119 15 137 17 83 11)
(4 112 124 42 140 180 75 224 109 200 24 138 234 81 64 198 77 170 103 120 232 54 61 35 88 38 6 139
127 205 51 141 154 208 214 52 34 32 168 157 126 69 143 100 202 131 176 184 129 230 190 209 25 31 115
41 169 47 7 29 5 165 237 242 108 67 116 97 39 223 53 87 148 125 95 175 128 16 30 222 163 206 104
173 21 218 26 84 228 244 215 105 147 235 217 212 185 182 22 111 231 161 99 66 226 216 79 37 195 160
46 196 50 167 183 236 28 191 23 164 20),

(2 3 4)(5 6 7)(8 9 10)(11 12 13)(14 15 16)(17 18 19)(20 21 22)(23 24 25)(26 27 28)(29 30 31)(32 33 34)
(35 36 37)(38 39 40)(41 42 43)(44 45 46)(47 48 49)(50 51 52)(53 54 55)(56 57 58)(59 60 61)(62 63 64)
(65 66 67)(68 69 70)(71 72 73)(74 75 76)(77 78 79)(80 81 82)(83 84 85)(86 87 88)(89 90 91)(92 93 94)
(95 96 97)(98 99 100)(101 102 103)(104 105 106)(107 108 109)(110 111 112)(113 114 115)(116 117 118)
(119 120 121)(122 123 124)(125 126 127)(128 129 130)(131 132 133)(134 135 136)(137 138 139)
(140 141 142)(143 144 145)(146 147 148)(149 150 151)(152 153 154)(155 156 157)(158 159 160)
(161 162 163)(164 165 166)(167 168 169)(170 171 172)(173 174 175)(176 177 178)(179 180 181)
(182 183 184)(185 186 187)(188 189 190)(191 192 193)(194 195 196)(197 198 199)(200 201 202)
(203 204 205)(206 207 208)(209 210 211)(212 213 214)(215 216 217)(218 219 220)(221 222 223)
(224 225 226)(227 228 229)(230 231 232)(233 234 235)(236 237 238)(239 240 241)(242 243 244),

(1 3 4)(5 153 124)(6 123 84)(7 83 154)(8 243 214)(9 213 164)(10 166 244)(11 36 34)(12 33 111)
(13 110 37)(14 181 155)(15 157 51)(16 50 179)(17 221 195)(18 194 41)(19 43 222)(20 63 61)
(21 60 218)(22 220 64)(23 145 116)(24 118 81)(25 80 143)(26 200 105)(27 104 71)(28 73 201)
(29 187 75)(30 74 120)(31 119 185)(32 35 112)(38 58 238)(39 237 100)(40 99 56)(42 196 223)
(44 199 192)(45 191 215)(46 217 197)(47 167 102)(48 101 177)(49 176 168)(52 156 180)(53 134 162)
(54 161 148)(55 147 135)(57 98 236)(59 62 219)(65 175 89)(66 91 107)(67 109 173)(68 205 230)
(69 232 211)(70 210 203)(72 106 202)(76 186 121)(77 139 141)(78 140 150)(79 149 137)(82 117 144)
(85 122 152)(86 97 94)(87 93 225)(88 224 95)(90 174 108)(92 96 226)(103 169 178)(113 171 188)
(114 190 182)(115 184 172)(125 159 131)(126 133 228)(127 227 160)(128 240 207)(129 206 235)
(130 234 241)(132 158 229)(136 146 163)(138 151 142)(165 212 242)(170 183 189)(193 198 216)
(204 209 231)(208 239 233),

(5 29 169 184 232)(6 30 167 182 230)(7 31 168 183 231)(8 56 90 93 135)(9 57 91 94 136)

(10 58 89 92 134)(11 17 71 150 241)(12 18 72 151 239)(13 19 73 149 240)(14 44 229 59 117)
(15 45 227 60 118)(16 46 228 61 116)(20 23 50 217 126)(21 24 51 215 127)(22 25 52 216 125)
(26 77 129 35 223)(27 78 130 36 221)(28 79 128 37 222)(32 196 105 141 235)(33 194 106 142 233)
(34 195 104 140 234)(38 175 226 53 244)(39 173 224 54 242)(40 174 225 55 243)(41 202 138 208 111)
(42 200 139 206 112)(43 201 137 207 110)(47 190 205 84 120)(48 188 203 85 121)(49 189 204 83 119)
(62 144 181 199 132)(63 145 179 197 133)(64 143 180 198 131)(65 96 162 166 238)(66 97 163 164 236)
(67 95 161 165 237)(68 123 74 102 114)(69 124 75 103 115)(70 122 76 101 113)(80 156 193 159 220)
(81 157 191 160 218)(82 155 192 158 219)(86 146 213 98 107)(87 147 214 99 108)(88 148 212 100 109)
(152 186 177 171 210)(153 187 178 172 211)(154 185 176 170 209).

Here are the block-orbit representatives of one 5-$(244, 6, 1)$ design, the indices give the order of the stabilizers of the orbits.

$\{1,2,3,4,5,243\}_2$	$\{1,2,3,5,41,130\}_2$	$\{1,2,3,6,7,147\}_2$
$\{1,2,3,4,11,61\}_2$	$\{1,2,3,5,42,52\}_2$	$\{1,2,3,6,11,172\}_2$
$\{1,2,3,5,6,88\}_2$	$\{1,2,3,5,44,81\}_2$	$\{1,2,3,6,19,85\}_2$
$\{1,2,3,5,9,169\}_2$	$\{1,2,3,5,45,175\}_2$	$\{1,2,3,6,22,210\}_2$
$\{1,2,3,5,12,66\}_2$	$\{1,2,3,5,46,47\}_2$	$\{1,2,3,6,32,223\}_2$
$\{1,2,3,5,13,35\}_2$	$\{1,2,3,5,53,218\}_2$	$\{1,2,3,6,33,103\}_2$
$\{1,2,3,5,14,38\}_2$	$\{1,2,3,5,54,121\}_2$	$\{1,2,3,6,36,140\}_2$
$\{1,2,3,5,15,60\}_2$	$\{1,2,3,5,55,120\}_2$	$\{1,2,3,6,38,98\}_2$
$\{1,2,3,5,16,30\}_2$	$\{1,2,3,5,64,180\}_2$	$\{1,2,3,6,39,66\}_2$
$\{1,2,3,5,17,242\}_2$	$\{1,2,3,5,69,131\}_2$	$\{1,2,3,6,45,236\}_2$
$\{1,2,3,5,18,167\}_3$	$\{1,2,3,5,70,157\}_2$	$\{1,2,3,6,47,145\}_2$
$\{1,2,3,5,19,190\}_2$	$\{1,2,3,5,77,156\}_2$	$\{1,2,3,6,56,87\}_2$
$\{1,2,3,5,20,85\}_2$	$\{1,2,3,5,78,236\}_2$	$\{1,2,3,6,74,178\}_2$
$\{1,2,3,5,21,61\}_2$	$\{1,2,3,5,86,100\}_2$	$\{1,2,3,6,89,146\}_2$
$\{1,2,3,5,22,186\}_2$	$\{1,2,3,5,90,154\}_2$	$\{1,2,3,6,206,221\}_2$
$\{1,2,3,5,25,89\}_2$	$\{1,2,3,5,105,210\}_2$	$\{1,2,3,7,18,43\}_2$
$\{1,2,3,5,27,228\}_2$	$\{1,2,3,5,111,199\}_2$	$\{1,2,3,7,34,141\}_2$
$\{1,2,3,5,28,68\}_2$	$\{1,2,3,5,123,192\}_2$	$\{1,2,3,7,35,145\}_2$
$\{1,2,3,5,29,219\}_2$	$\{1,2,3,5,125,216\}_2$	$\{1,2,3,7,37,74\}_5$
$\{1,2,3,5,33,104\}_2$	$\{1,2,3,5,155,203\}_2$	$\{1,2,3,7,53,207\}_2$
$\{1,2,3,5,34,138\}_5$	$\{1,2,3,5,166,217\}_2$	$\{1,2,3,7,102,214\}_5$
$\{1,2,3,5,36,168\}_2$	$\{1,2,3,5,187,222\}_2$	$\{1,2,3,11,51,137\}_5$
$\{1,2,3,5,37,99\}_2$		

For the solving of the Diophantine linear systems we implemented a solver after Mathon's algorithm `spreads` [9], as part of DISCRETA. The first 5-$(244, 6, 1)$ design with automorphism group $P\Sigma L(2, 3^5)$ was found with a randomized version of this algorithm by the third author.

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