A Steiner 5-Design on 36 Points

Anton Betten, Reinhard Laue, Alfred Wassermann
Mathematical Department, University of Bayreuth, Germany

June 9, 1998

Abstract *All known Steiner 5-designs previously had an order $q+1$ where $q \equiv 3 \pmod 4$ is a prime power and $PSL(2,q)$ was admitted as a group of automorphisms of such a design. In this paper we present a 5-$(36,6,1)$ design admitting $PGL(2,17) \times C_2$ as a group of automorphisms. The design is unique with this automorphism group and even for the commutator group $PSL(2,17) \times id_2$ of this automorphism group there exists no further design with these parameters. We list the intersection numbers of this Steiner system and show the incidence matrix of t-subset orbits and block orbits, to allow an analysis of the design.*

1 Introduction

For a long time t-designs were known only for $t \leq 5$ and these designs admitted some group $PSL(2,q)$ as a group of automorphisms. The full automorphism group could be larger as in the case of the famous Witt designs [20]. Assmus and Mattson [1] contributed such designs for the cases $q = 23,\ 48$ deriving them from codes. The new designs had values of λ greater than 1, and in several cases consisted of just one orbit of the group $PSL(2,q)$. The search for t-designs at that time was closely related to the search for transitive extensions of permutation groups [14]. So, Assmus and Mattson pointed out that their new 5-designs are not "orbit-designs", i.e. $PSL(2,q)$ is not transitive on the set of blocks of the design. A few years later, Denniston [10], [9], Mills [18], Grannell, Griggs [11], constructed Steiner 5-systems, i.e. 5-$(q+1,k,1)$ designs, where they used some $PSL(2,q)$ as a prescribed group of automorphisms too. Denniston also noticed that several orbits of the group on k-subsets had to be combined to obtain such a design. More generally, using the classification theorem of finite simple groups, Praeger and Cameron [7] showed that no block-transitive 8-(v,k,λ) designs exist and conjectured that already block-transitive 6-(v,k,λ) designs do not exist. Therefore it is a natural requirement to have a tool that constructs t-designs by combining many orbits. This has been formalized nicely by Kramer and Mesner [15]. The authors have developed a software package DISCRETA which has produced already several 6-, 7-, and 8-designs by this approach. Details may be found in the papers [19], [6], [4].

A difficult task in finding new t-designs by a software tool like DISCRETA still is to predict a group of automorphisms of the designs. So, since all Steiner 5-designs known previously have an order $q+1$ where $q \equiv 3 \pmod 4$ is a prime power and $PSL(2,q)$ is admitted as a group of automorphisms [8], Denniston [10], Grannell, Griggs, Mathon [12], [13], [21], and Mathon [17] looked for further values of q to construct new Steiner 5-systems. In this paper we slightly modify the permutation presentation of $PSL(2,17)$ and thus act on a set of v points, where $v-1$ is not a prime power. So, we present a simple 5-$(36,6,1)$ design admitting $PGL(2,17) \times C_2$ as a group of automorphisms. The design is unique with this automorphism group and even prescribing only the commutator group $PSL(2,17) \times Id_2$ of this automorphism group yields no further simple design with these parameters. We list some characteristics of this Steiner system and the incidence matrix of t-subset orbits and block orbits, to allow an analysis of the design. DISCRETA also shows that with this prescribed automorphism group there exist simple 5-$(36,6,\lambda)$ designs for each of the admissible values of $\lambda = 1,2,\cdots,31$.

Readers interested in DISCRETA may consult our WWW-page listing many t-designs especially for $t \geq 6$ and allowing to download DISCRETA for various platforms:
http://mathe2.uni-bayreuth.de/betten/DESIGN/d1.html

We remark that this kind of direct product groups has already lead to other important t-designs. According to A. Brouwer [3], Denniston [9] has found a 4-(12,5,4) design with $PGL(2,5) \times Id_2$, S. Bays and E. De Weck [2] found a 3-(14,4,1) already in 1935, using $Hol(C_7)_2 \times Id_2$. Here $Hol(C_7)_2$ is the subgroup of index 2 of the holomorph of C_7. A. Brouwer [3] used the latter group to get some 3-(15,5,λ) designs for different values of λ, a 4-(15,5,3) design, and using $PSL(2,7) \times Id_2$ many 3-(16,4,λ) designs for different values of λ. Also 5-(16,6,3) and 5-(16,6,5) designs arouse from this group. A 4-(18,8,84) design came from a $PGL(2,8) \times C_2$.

We have first used $PGL(2,9) \times Id_2$ to find large sets $LS[2](4,9,20)$, $LS[2](4,10,20)$, and adding a fixed point to $PSL(2,9) \times Id_2$ we found an $LS[2](5,9,21)$. $PSL(3,2) \times S_3$ acting with an additional fixed point gave an $LS[2](6,9,22)$[16]. While in the case of an $LS[2](t,k,v)$ the value of λ in the t-designs is as large as possible up to complementary designs, in this paper we are interested in the smallest possible value $\lambda = 1$. We have obtained some cases where $\lambda = 2$ or λ is smallest admissible. So far the above Steiner system is the only one we could derive from this kind of groups. We list some parameter sets and the direct product groups used to find the designs. Some of these parameter sets are new, others had already been found using other types of groups.

Parameter	Group
7-(20,10,126)	$(PSL(2,8) \times C_2) ++$ [5]
4-(16,6,6)	$(D_5 \times C_3)+$ [5]
3-(14,5,5)	$(C_7 \times C_2)$ [5]
5-(24,6,2)	$PSL(2,11) \times C_2$
5-(24,8,288)	$(M_{11})_{12} \times Id_2$
5-(24,9,1080)	$(M_{11})_{12} \times Id_2$
5-(28,6,2)	$PGL(2,13) \times C_2$
5-(34,6,5)	$P\Gamma L(2,16) \times C_2$
5-(36,6,1)	$PGL(2,17) \times C_2$
5-(46,8,800)	$M_{23} \times Id_2$
5-(48,6,2)	$PGL(2,23) \times C_2$
5-(52,6,2)	$P\Gamma L(2,25) \times C_2$

In this table, $(M_{11})_{12}$ denotes the permutation representation of the Mathieu group M_{11} on 12 points.

2 Steiner systems with prescribed automorphisms

In order to explain the choice of blocks which gives the $5\text{-}(36,6,1)$ design we make some remarks on the connection between the design and its automorphism group.

Lemma. *Let G be a group of automorphisms of a t-$(v,k,1)$ Steiner system $\mathcal{D}$. Then the mapping $\phi : \binom{V}{t} \to \mathcal{D}$ mapping each t-subset of the underlying point set V onto the unique block B containing T commutes with the actions of G on $\binom{V}{t}$ and $\mathcal{D}$. In particular, for each $T \in \binom{V}{t}$ its stabilizer $N_G(T)$ is contained in $N_G(\phi(T))$, the stabilizer of the block containing T. Any two t-subsets of a block B which are in the same orbit under G must be in the same orbit under $N_G(B)$.*

The 6-(36,6,1) design consists of 15 orbits of blocks under the action of $PGL(2,17) \times C_2$. It is easily veryfied that for each block-orbit representative B the sum over all $|N_G(B)|/|N_G(T)|$ for $T \subset B$ a representative of a 5-orbit is 6, as required by the Lemma. The block orbits can be classified with respect to the orders of the stabilizers in G. The structure of the group action allows a refinement by the number of points belonging to the same orbit of $PGL(2,17)$. So, a 5-subset has two numbers assigned counting the entries from $\{1, \cdots, 18\}$ and from $\{19, \cdots, 36\}$. Any $g \in G$ applied to B may interchange these numbers, but fixes the set of these numbers. So, we draw a bipartite graph with the block orbits and the 5-orbits as vertices. If a block B contains m 5-subsets from the orbit of T then we connect the vertices for B^G and T^G by an edge of multiplicity m.

3 The 5-$(36, 6, 1)$ design

The prescribed group of automorphisms has order 9792 and is isomorphic to $PGL(2, 17) \times C_2$. Its permutation representation is obtained by taking two copies of the natural point set of $PGL(2, 17)$ and the same action of this group on both sets. A permutation of order two interchanging the corresponding points is taken as an additional generator of the group. Thus we obtain the following presentation by generating permutations.

$G =$

$$\begin{gathered}\langle (1\,19)(2\,20)(3\,21)(4\,22)(5\,23)(6\,24)(7\,25)(8\,26)(9\,27)(10\,28)(11\,29)(12\,30)(13\,31)(14\,32)\\ (15\,33)(16\,34)(17\,35)(18\,36),\\ (3\,5\,11\,12\,15\,7\,17\,13\,18\,16\,10\,9\,6\,14\,4\,8)(21\,23\,29\,30\,33\,25\,35\,31\,36\,34\,28\,27\,24\,32\,22\,26),\\ (3\,8\,4\,14\,6\,9\,10\,16\,18\,13\,17\,7\,15\,12\,11\,5)(21\,26\,22\,32\,24\,27\,28\,34\,36\,31\,35\,25\,33\,30\,29\,23),\\ (2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14\,15\,16\,17\,18)(20\,21\,22\,23\,24\,25\,26\,27\,28\,29\,30\,31\,32\,33\,34\,35\,36),\\ (1\,3\,11\,8\,15\,9\,5\,7\,17\,4\,14\,16\,12\,6\,13\,10\,18)(19\,21\,29\,26\,33\,27\,23\,25\,35\,22\,32\,34\,30\,24\,31\,28\,36)\rangle\end{gathered}$$

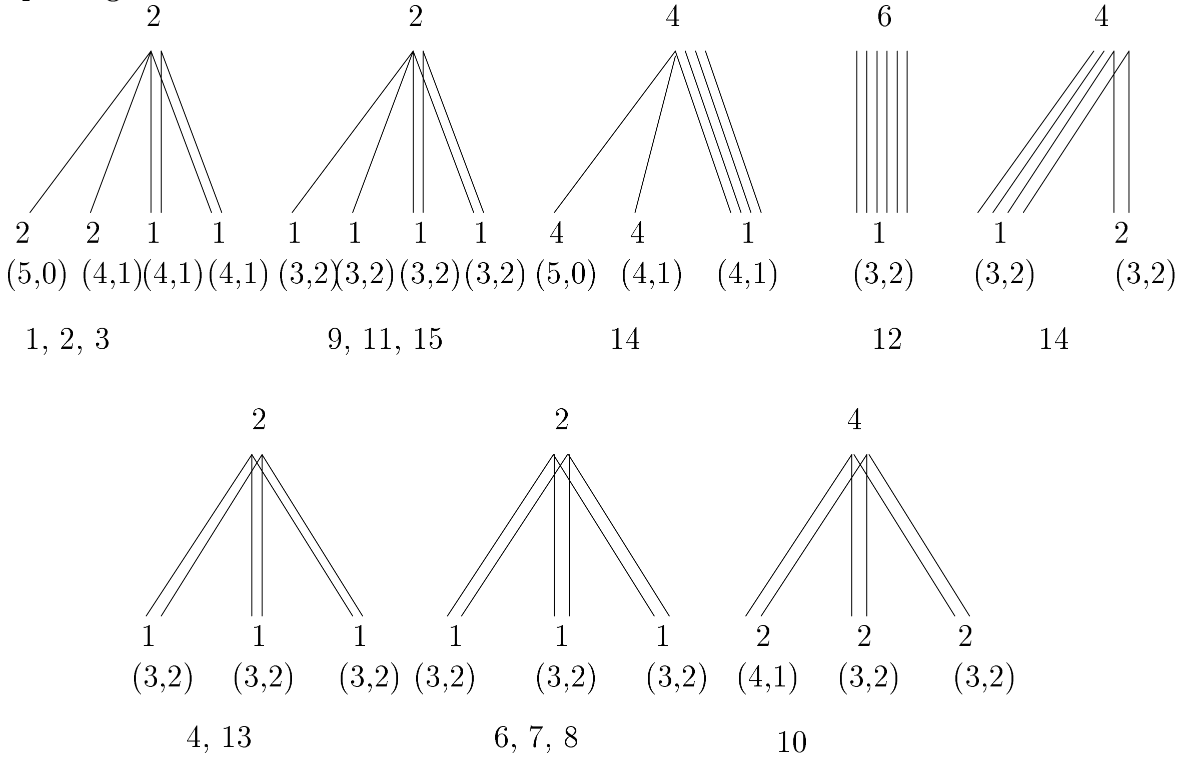
There are 48 orbits on 5-sets which are shown in the second column of the next table. The Steiner system consists of 15 6-orbits out of 259 orbits. A block of the Steiner system can be described by a 5-subset and the additional point in the block. So, the first 15 rows of the table belong to 5-orbit representatives which are to be extended by one additional element to obtain the unique block of the design that contains that 5-set. These elements are listed as additional points above the inclusion matrix. Other 5-subsets of a block will belong to further orbits which can be found in the following rows.

The inclusion relation of 5-orbits and blocks (size 48×15)

orbit number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
stabilizer order	2	2	2	2	4	2	2	2	2	4	2	6	2	4	2
additional point	25	30	26	30	34	24	28	35	36	21	19	31	32	31	33
5-orbits															
$\{1,2,3,4,8\}_2$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,5\}_2$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,9\}_2$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,6,26\}_1$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,7\}_4$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,22,23\}_1$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\{1,2,3,22,25\}_1$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\{1,2,3,22,31\}_1$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$\{1,2,3,19,22\}_1$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\{1,2,3,4,19\}_2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{1,2,3,19,23\}_1$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{1,2,3,24,25\}_1$	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
$\{1,2,3,5,27\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\{1,2,3,19,27\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$\{1,2,3,19,25\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\{1,2,3,6,19\}_1$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,26\}_2$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,19\}_1$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,6,22\}_2$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,22\}_1$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,23\}_1$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,6,23\}_1$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,24\}_1$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,5,28\}_2$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,23,33\}_1$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,23,28\}_1$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,6,25\}_1$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,4,25\}_4$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$\{1,2,3,22,24\}_1$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\{1,2,3,23,24\}_1$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\{1,2,3,22,26\}_1$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\{1,2,3,23,31\}_1$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\{1,2,3,22,32\}_1$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$\{1,2,3,23,27\}_1$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
$\{1,2,3,22,29\}_2$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\{1,2,3,23,30\}_2$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\{1,2,3,19,26\}_1$	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
$\{1,2,3,19,29\}_2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{1,2,3,19,20\}_2$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{1,2,3,24,34\}_2$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{1,2,3,19,24\}_1$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{1,2,3,23,35\}_2$	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{1,2,3,23,25\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\{1,2,3,23,34\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\{1,2,3,24,33\}_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$\{1,2,3,24,27\}_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\{1,2,3,23,26\}_2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\{1,2,3,19,28\}_1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

The index appended to a 5-orbit representative denotes the order of the stabilizer of that 5-subset.

Splitting of blocks into 5-orbits



In the first row the stabilizer orders of the block orbits are shown. The bipartite graph relates block orbits to 5-orbits. An edge of multiplicity m means that m 5-sets of the 5-orbit are contained in the same block from the block orbit. The first row below the graph denotes the stabilizer orders of the 5-orbits. The second row below the graph denotes the split type of the 5-orbit, i. e. the distribution of points to the two sets of 18 points. The third row below the graph denotes the orbit numbers of the blocks which fit to the tree pattern above the numbers.

Intersection Numbers

b = 62832
r = 10472
The Mendelsohn system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 & 5 & 6 \\ & & 1 & 3 & 6 & 10 & 15 \\ & & & 1 & 4 & 10 & 20 \\ & & & & 1 & 5 & 15 \\ & & & & & 1 & 6 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \begin{pmatrix} 62832 \\ 62832 \\ 22440 \\ 3520 \\ 240 \\ 6 \end{pmatrix} = \begin{pmatrix} \binom{6}{0} & 62832 \\ \binom{6}{1} & 10472 \\ \binom{6}{2} & 1496 \\ \binom{6}{3} & 176 \\ \binom{6}{4} & 16 \\ \binom{6}{5} & 1 \end{pmatrix} \tag{1}$$

The (unique) solution:

$$\begin{pmatrix} 19155 \\ 27576 \\ 13275 \\ 2600 \\ 225 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

The triangle

$$
\begin{array}{rrrrrr}
62832 & 52360 & 43384 & 35728 & 29232 & 23751 \\
10472 & 8976 & 7656 & 6496 & 5481 & \\
1496 & 1320 & 1160 & 1015 & & \\
176 & 160 & 145 & & & \\
16 & 15 & & & & \\
1 & & & & &
\end{array} \tag{3}
$$

References

[1] E. F. Assmus, H. F. Mattson: New 5-designs. *J. Comb. Theory* **6** (1969), 122-151.

[2] S. Bays, E. De Weck: Sur les systèmes de quadruples. *Comment. Math. Helvet.* **7** (1935), 222-241.

[3] A. E. Brouwer: Table of t-designs without repeated blocks, $2 \leq t \leq k \leq v/2, \lambda \leq \lambda^{+}/2$. *Math. Centrum. Report ZN76*, Amsterdam 1977, unpublished update 1986.

[4] A. Betten, R. Laue, A. Wassermann: Some simple 7-designs. to appear in *J. Comb. Designs.*

[5] A. Betten, R. Laue, A. Wassermann: Designs on up to 21 points, some difficult cases. in preparation.

[6] A. Betten, A. Kerber, R. Laue, A. Wassermann: Simple 8-designs with small parameters. to appear in *J. Designs, Codes and Cryptography.*

[7] P. J. Cameron, C. E. Praeger: Block-transitive t-designs I: point-imprimitive designs. *Discr. Math.* **118** (1993), 33–43.

[8] C. J. Colbourn, R. Mathon: Steiner systems. C. J. Colbourn, J. H. Dinitz, ed., *The CRC Handbook of Combinatorial Designs,*66–75, CRC Press, Boca Raton (1996).

[9] R. H. F. Denniston: A small 4-design. *Ann. Discrete Math.* **18** (1983), 291–294.

[10] R. H. F. Denniston: Some new 5-designs. *Bull. London Math. Soc.* **8** (1976), 263–267.

[11] M. J. Grannell, T. S. Griggs: On Steiner systems $S(5,6,24)$. *Ars Combinatoria* **8** (1979), 45–48.

[12] M. J. Grannell, T. S. Griggs, R. Mathon: Some Steiner 5-designs with 108 and 132 points. *J. Comb. Designs* **1** (1993), 213–238.

[13] M. J. Grannell, T. S. Griggs, R. Mathon: Steiner Systems $S(5,6,v)$ with $v = 72$ and 84. *Mathematics of Computation* **67** (1998), 357–359.

[14] D. R. Hughes: On t-designs and groups. *Amer. J. Math.* **87** (1965), 761–778.

[15] E. S. Kramer, D. Mesner: t-designs on hypergraphs. *Discr. Math.* **15** (1976), 263–296.

[16] R. Laue: Halvings on small point sets. Preprint: *Center of Discrete Mathematics and Theoretical Computer Science* Technical Report **CDMTCS-078**, University of Auckland, New-Zealand. Extended version is to appear in Journal of Comb. Designs.

[17] R. Mathon: Searching for spreads and packings. *in Geometry, Combinatorial Designs and Related Structures, Proceedings of the first Pythagorean conference* J. W. P. Hirschfeld, S. S. Magliveras, M. J. de Resmini ed., *London Math. Soc. LN* **245** (1997), 161–176.

[18] W. H. Mills: A new 5-design. *Ars Combinatoria* **6** (1978), 193–195.

[19] A. Wassermann: Finding simple t-designs with enumeration techniques. *J. Combinatorial Designs* **6** (1998), 79–90.

[20] E. Witt: Über Steinersche Systeme. *Abh. Math. Sem. Hamburg* **12** (1938), 265–275.

[21] B. Schmalz: t-Designs zu vorgegebener Automorphismengruppe. *Dissertation, Univ. Bayreuth* (1992).