# Genealogy of $t$-designs I: ancestors and families

Anton Betten, Evi Haberberger, Reinhard Laue

June 29, 2001

**Abstract**

For a long time, there as been interest in finding $t$-designs with large $t$ on a small number of points. In the last few years, existence results in that area have mushroomed. Todays knowledge amounts to several thousand parameter sets of such designs, even when restricting to $t \geq 5$. The current work contributes to the classification of parameter sets of $t$-designs. We are primarily interested in $t$-designs with large $t$ but most of the work presented here applies to the general case of arbitrary $t$. Given an admissible parameter set of an incomplete $t$-design, we show that there is a largest possible admissible parameter set from which the given one can be obtained by reducing $t$, forming the parameter set of a residual or a derived design. This unique parameter set is called an ancestor parameter set. An ancestor comes together with a whole family of $t$-design parameter sets, some of which may be realizable, others may not. We characterize the parameter sets contained in a family. In a special case of Steiner systems, we evaluate the ancestor directly. As a final application, we look at the particular family generated by 15-$(32, 16, 6)$ and extend a lower bound for the number of isomorphism types of the 7-designs in that family. In a successing paper, we will examine the ancestor parameter sets more deeply. This study eventually leads to a compact classification of $t$-design parameter sets with large $t$.

Keywords: t-design, parameter set, ancestor, family, isomorphism types.

## 1 Introduction

A $t$-$(v, k, \lambda)$ design is a pair $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a set of $v$ elements called points and $\mathcal{B} = \{B_1, \ldots, B_b\}$ is a set of $k$-subsets of $\mathcal{V}$ – called blocks – such that each $t$-element subset of $\mathcal{V}$ is contained in exactly $\lambda$ blocks. The quadruple of integers $t$-$(v, k, \lambda)$ is called the parameter set of the design. The integer $t$ is the *point regularity*, $v$ is the *size of the underlying point set*, $k$ is the *block size* and $\lambda$ is the *index.* Sometimes, even more natural numbers are considered to be the parameters of a design. For instance, the *number of blocks* $b$ is determined by $t, v, k$ and $\lambda$.

An *isomorphism* of a design $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ onto another design $\mathcal{D}' = (\mathcal{V}', \mathcal{B}')$ is a bijection $\phi : \mathcal{V} \to \mathcal{V}'$ which by replacing the points of each block of $\mathcal{B}$ by their image points maps the blocks of $\mathcal{D}$ onto the blocks of $\mathcal{D}'$. An isomorphism can be understood as being just a renaming of the points. An *automorphism* of a design $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ is an isomorphism of the design onto itself. The set of automorphisms forms a group, the *automorphism group*, denoted by $\mathrm{Aut}(\mathcal{D})$. A design with $\lambda = 1$ is called *Steiner System.* Certain designs are so obvious that one considers them as trivial. One is the *complete design* which consists of *all* $k$-subsets. It is a $t$-design for all $t \leq k$. The parameters as a $k$-design are $k$-$(v, k, 1)$, with $b = \binom{v}{k}$.

Let $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ be a $t$-$(v, k, \lambda)$ design with $1 \leq t \leq k < v$. Then $\mathcal{D}$ yields further designs:

(i) The design $\mathcal{D}$ also is a $(t-1)$-$(v, k, \lambda_{\text{red}})$ design where $\lambda_{\text{red}} = \frac{v-t+1}{k-t+1}$. We call it the design with respect to smaller $t$, or simply the *reduced design* $\operatorname{red} \mathcal{D}$.

(ii) If $x$ is a point of $\mathcal{V}$, the *derived design* (with respect to the point $x$) is $\operatorname{der}_x \mathcal{D} = (\mathcal{V}\backslash\{x\}, \mathcal{B}_x)$ where $\mathcal{B}_x = \{B\backslash\{x\} \mid B \in \mathcal{B},\ x \in B\}$. Each derived design has parameters $(t-1)$-$(v-1, k-1, \lambda)$ (regardless of the choice of the point $x \in \mathcal{V}$). Put $\lambda_{\text{der}} = \lambda$.

(iii) If $x$ is a point of $\mathcal{V}$, the *residual design* (with respect to the point $x$) is $\operatorname{res}_x \mathcal{D} = (\mathcal{V}\backslash\{x\}, \mathcal{B}^x)$ where $\mathcal{B}^x = \{B \mid B \in \mathcal{B},\ x \notin B\}$. Each residual design has parameters $(t-1)$-$(v-1, k, \lambda_{\text{res}})$ where $\lambda_{\text{res}} = \lambda \frac{v-k}{k-t+1}$ (regardless of the choice of the point $x \in \mathcal{V}$). For the purpose of forming the residual design, the assumption $k < v$ is required.

Note that

$$\lambda_{\text{red}} = \lambda_{\text{der}} + \lambda_{\text{res}} = \lambda + \lambda_{\text{res}}. \tag{1}$$

The isomorphism type of the resulting design depends on the choice of the points we take while forming the residual or the derived design. Since the parameter sets which we obtain are independent of this choice, we define the operators red, der and res in a more abstract way, namely as maps between parameter quadruples: For $t$-$(v, k, \lambda)$, we let $\operatorname{red} t$-$(v, k, \lambda)$, $\operatorname{der} t$-$(v, k, \lambda)$ and $\operatorname{res} t$-$(v, k, \lambda)$ be the parameter set of the reduced, the derived and the residual design, respectively. The operators red, der and res on parameter sets commute. Thus, for nonnegative integers $h$, $i$ and $j$ with $h + i + j \leq t$ and $j \leq v - k$, the parameter set $\operatorname{red}^h \operatorname{der}^i \operatorname{res}^j t$-$(v, k, \lambda)$ is obtained from $t$-$(v, k, \lambda)$ by reducing $h$ times, deriving $i$ times and forming the residual $j$ times.

## 2 Admissibility of parameter sets

Not every quadruple of nonnegative integers is a valid parameter set of a design. Some restrictions can be derived from number theoretic constraints. Another is due to the fact that the number of blocks is bounded above by the number of $k$-subsets. We collect these necessary conditions in the following definition. We explicitely state that the parameters have to be nonnegative integers since later on we are considering putative parameter sets in the rational numbers.

**Definition 2.1** Let $t$, $v$, $k$ and $\lambda$ be nonnegative rationals. The parameter set $t$-$(v, k, \lambda)$ is called *admissible* if (ADM0), (ADM1), (ADM2) and (ADM3-$s$) hold for $0 \leq s \leq t$ where

(ADM0) $t$-$(v, k, \lambda)$ is integral, i. e. $t$, $v$, $k$ and $\lambda$ are integers,

(ADM1) $t \leq k \leq v$,

(ADM2) $1 \leq \lambda \leq \binom{v-t}{k-t}$,

(ADM3-$s$) $\lambda\frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}} = \lambda\frac{(v-s)(v-s-1)\cdots(v-t+1)}{(k-s)(k-s-1)\cdots(k-t+1)} = \lambda_{\mathrm{red}^{t-s}}$ is integral.

(ADM3-$t$) merely states that $\lambda$ is integral (which is also required for (ADM0)). (ADM3-0) implies that the "number of blocks" is an integer (there need not be a design, but the formula for the number of blocks evaluates to an integer). Put

$$\lambda_{\max}(t, v, k) := \binom{v-t}{k-t} \tag{2}$$

so that (ADM2) becomes $1 \leq \lambda \leq \lambda_{\max}(t, v, k)$. If the parameter set is understood, we simply write $\lambda_{\max}$ instead of $\lambda_{\max}(t, v, k)$. Note that $\lambda_{\max}$ is the maximal index a $t$-design on $v$ points with block size $k$ can have, therefore the name:

**Lemma 2.2** *If $t$-$(v, k, \lambda)$ is admissible then $\lambda \leq \lambda_{\max}(t, v, k)$.*

*Proof:*

$$\begin{aligned} & \frac{v\cdots(v-k+1)}{k\cdots 1} = \binom{v}{k} \geq b = \lambda\frac{\binom{v}{t}}{\binom{k}{t}} = \lambda\frac{v\cdots(v-t+1)}{k\cdots(k-t+1)} \\ \iff \lambda_{\max} = \binom{v-t}{k-t} = & \frac{(v-t)\cdots(v-k+1)}{(k-t)\cdots 1} \geq \lambda. \end{aligned}$$

□

Hence, the second inequality in (ADM2) is derived from the fact that the blocks are $k$-subsets. A parameter set of the form $t$-$(v, k, \lambda_{\max})$ is called *complete*, as it describes the complete design. Hence for fixed $t$, $v$ and $k$ with $t \leq k \leq v$ there always is at least one admissible parameter set. The following result is straightforward.

**Lemma 2.3** *Let $D\big(t\text{-}(v,k,\lambda)\big) = t'\text{-}(v',k',\lambda')$ be admissible for some $D \in \{\mathrm{red},\mathrm{der},\mathrm{res}\}$ and admissible $t\text{-}(v,k,\lambda)$. Then*

$$\frac{\lambda}{\lambda_{\max}(t,v,k)} = \frac{\lambda'}{\lambda_{\max}(t',v',k')}.$$

*In other words, the function $\lambda_{\max}$ is proportional to the index under the three operations. In particular, $D\big(t\text{-}(v,k,\lambda)\big)$ is complete if and only if $t\text{-}(v,k,\lambda)$ is complete. A further consequence is that the condition (ADM2) for a parameter set $t\text{-}(v,k,\lambda)$ and for $D\big(t\text{-}(v,k,\lambda)\big)$ coincide.*

**Lemma 2.4** *If $t\text{-}(v,k,\lambda)$ is admissible with $t \geq 1$, then $\mathrm{red}\, t\text{-}(v,k,\lambda)$ and $\mathrm{der}\, t\text{-}(v,k,\lambda)$ are admissible, too. In addition, if $k < v$, then $\mathrm{res}\, t\text{-}(v,k,\lambda)$ is admissible as well.*

*Proof:* By Lemma 2.3, $1 \leq \lambda \leq \lambda_{\max}(t,v,k)$ implies $1 \leq \lambda' \leq \lambda_{\max}(t',v',k')$ for $D\big(t\text{-}(v,k,\lambda)\big) = t'\text{-}(v',k',\lambda')$ and $D \in \{\mathrm{red},\mathrm{der},\mathrm{res}\}$, so (ADM2) is satisfied in all cases. Let us consider the remaining conditions of admissibility:

- The parameters of $\mathrm{red}\, t\text{-}(v,k,\lambda) = (t-1)\text{-}\Big(v,k,\lambda\frac{v-(t-1)}{k-(t-1)}\Big)$ are integral by $\big(\text{ADM3-}(t-1)\big)$. Moreover, by induction the parameters of iterated reduced designs are integral as well: $\mathrm{red}^i \mathrm{red}\, t\text{-}(v,k,\lambda) = \mathrm{red}^{i+1}\, t\text{-}(v,k,\lambda)$ is admissible for $i \leq t-1$. Finally, $t-1 \leq k \leq v$.

- $\mathrm{der}\, t\text{-}(v,k,\lambda) = (t-1)\text{-}(v-1,k-1,\lambda)$. For $0 \leq i \leq t-1$, $\mathrm{red}^i\, \mathrm{der}\, t\text{-}(v,k,\lambda) = \mathrm{der}\,\mathrm{red}^i\, t\text{-}(v,k,\lambda)$ is integral. (ADM1) is valid since $t-1 \leq k-1 \leq v-1$.

- If $k < v$, the operator res is defined. We prove that $\mathrm{res}\, t\text{-}(v,k,\lambda) = (t-1)\text{-}\Big(v-1,k,\lambda\frac{v-k}{k-(t-1)}\Big)$ is admissible. Using (1) we get $\lambda_{\mathrm{res}} = \lambda\frac{v-k}{k-(t-1)} = \lambda_{\mathrm{red}} - \lambda$ is integral. In addition, $t-1 \leq k \leq v-1$. □

In order to describe all admissible parameter sets, we introduce the number

$$\Delta\lambda(t,v,k) := \mathrm{lcm}\left\{ \frac{\binom{k-s}{t-s}}{\gcd\left(\binom{k-s}{t-s},\binom{v-s}{t-s}\right)} \;\middle|\; 0 \leq s \leq t \right\}. \tag{3}$$

Then

$$\Delta\lambda(t,v,k) = \Delta\lambda(t,v,v-k), \tag{4}$$

$$\Delta\lambda(k,v,k) = \Delta\lambda(0,v,k) = \Delta\lambda(t,v,v) = 1 \tag{5}$$

for all $t \leq k \leq v$. It turns out that for fixed $t$, $v$ and $k$, $\Delta\lambda(t,v,k)$ is the smallest value of $\lambda$ for an admissible parameter set $t\text{-}(v,k,\lambda)$. The next result characterizes admissible parameter sets:

**Proposition 2.5** *Let $t, v, k$ and $\lambda$ be nonnegative rationals with $k < v$. The following conditions are equivalent:*

*(i)* *The parameter set $t\text{-}(v, k, \lambda)$ is admissible.*

*(ii)* $\mathrm{red}^i\, t\text{-}(v, k, \lambda)$ *is integral for $0 \le i \le t$, where $t \le k$ and $1 \le \lambda \le \lambda_{\max}(t, v, k)$.*

*(iii)* $t, v, k, \lambda \in \mathbb{N}$, $\Delta\lambda(t, v, k) \mid \lambda$, $t \le k \le v$ *and* $1 \le \lambda \le \lambda_{\max}(t, v, k)$.

*(iv)* $\mathrm{der}\, t\text{-}(v, k, \lambda)$ *and* $\mathrm{res}\, t\text{-}(v, k, \lambda)$ *are admissible.*

*(v)* $\mathrm{red}\, t\text{-}(v, k, \lambda)$ *and* $\mathrm{der}\, t\text{-}(v, k, \lambda)$ *are admissible.*

*(vi)* $\mathrm{red}\, t\text{-}(v, k, \lambda)$ *and* $\mathrm{res}\, t\text{-}(v, k, \lambda)$ *are admissible and $t' < k'$ where* $\mathrm{res}\, t\text{-}(v, k, \lambda) = t'\text{-}(v', k', \lambda')$.

*Proof:*

$(i) \Leftrightarrow (ii)$ : As $\mathrm{res}^i\, t\text{-}(v, k, \lambda) = s\text{-}\left(v, k, \lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}\right)$ with $s := t - i$, $(i)$ and $(ii)$ are equivalent.

$(i) \Leftrightarrow (iii)$ : If $t\text{-}(v, k, \lambda)$ is admissible, then for $0 \le s \le t$, the number $\lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$ is integral. Thus $\binom{k-s}{t-s}$ divides $\lambda \binom{v-s}{t-s}$ which implies that $\frac{\binom{k-s}{t-s}}{\gcd\left(\binom{k-s}{t-s}, \binom{v-s}{t-s}\right)}$ divides $\lambda$ for all these $s$. Therefore $\Delta\lambda$ divides $\lambda$. The other implication is clear.

$(i) \Leftrightarrow (iv)$ : By Lemma 2.4, $(i)$ implies $(iv)$. On the other hand, assume that $\mathrm{der}\, t\text{-}(v, k, \lambda)$ and $\mathrm{res}\, t\text{-}(v, k, \lambda)$ are admissible. In particular, $\lambda_{\mathrm{res}} = \lambda \frac{v-k}{k-(t-1)}$ is integral. Then (1) implies $\lambda_{\mathrm{red}} = \lambda_{\mathrm{der}} + \lambda_{\mathrm{res}}$ is integral, so $\mathrm{red}\, t\text{-}(v, k, \lambda)$ is integral. It remains to show that $\lambda_{\mathrm{red}^i}$ is integral for $1 < i \le t$. If $t = 1$, there is nothing to show. So assume $t \ge 2$. We apply (1) to get $\lambda_{\mathrm{red}^2} = (\lambda_{\mathrm{red}})_{\mathrm{der}} + (\lambda_{\mathrm{red}})_{\mathrm{res}} = (\lambda_{\mathrm{der}})_{\mathrm{red}} + (\lambda_{\mathrm{res}})_{\mathrm{red}}$, parameters which are integral by our assumption. So $\lambda_{\mathrm{red}^2}$ is integral. We can proceed by induction.

$(i) \Leftrightarrow (v)$ : By Lemma 2.4, $(i)$ implies $(v)$. Now assume that $\mathrm{red}\, t\text{-}(v, k, \lambda)$ and $\mathrm{der}\, t\text{-}(v, k, \lambda)$ are admissible. In particular, $\lambda_{\mathrm{der}} = \lambda$ is integral which is (ADM3-$t$). Moreover, $t \le k \le v$ since $t - 1 \le k - 1 \le v - 1$ holds in the derived parameter set. And $\lambda \le \lambda_{\max}(v, t, k)$ by Lemma 2.3. The conditions (ADM3-$s$) for $0 \le s \le t - 1$ are satisfied, hence $t\text{-}(v, k, \lambda)$ is admissible.

$(i) \Leftrightarrow (vi)$ : Again by Lemma 2.4, $(i)$ implies $(vi)$. Now assume that $\mathrm{red}\, t\text{-}(v, k, \lambda)$ and $\mathrm{res}\, t\text{-}(v, k, \lambda)$ are admissible. By (1), $\lambda = \lambda_{\mathrm{der}} = \lambda_{\mathrm{red}} - \lambda_{\mathrm{res}}$ is integral which is (ADM3-$t$). The assumption $t' < k'$ implies $t - 1 < k$, hence $t \le k < v$. (ADM2) follows by Lemma 2.3. The conditions (ADM3-$s$) for $0 \le s \le t - 1$ correspond to the conditions for the parameter set $\mathrm{red}\, t\text{-}(v, k, \lambda)$ for $s > 0$.

$\square$

A parameter set $t$-$(v,k,\lambda)$ is called *realizable* if there exists a design with these parameters. In particular, realizable parameters are admissible. Since the complete design exists, we derive from Proposition 2.5, (i) $\Leftrightarrow$ (iii), that $\Delta\lambda(t,v,k)$ divides $\lambda_{\max}(t,v,k)$.

There exist admissible parameter sets which are known not to be realizable: For example, there is no 13-$(32,16,3)$ design as shown by Köhler [6].

It may happen that a given design is the derived or residual of another design. Also the point regularity of the design may be increased, i. e. a given $t$-design may turn out to be a $(t+1)$-design. Of course, the parameters of the given design must satisfy additional properties for this to be possible. We are led to introduce inverse operators $\mathrm{red}^{-1}$, $\mathrm{der}^{-1}$ and $\mathrm{res}^{-1}$ in such a way that the following equivalences for admissible $t$-$(v,k,\lambda)$ with $t \geq 1$ hold:

$$\begin{aligned} & \mathrm{red}\big(t\text{-}(v,k,\lambda)\big) &=& \quad (t-1)\text{-}\big(v,k,\lambda\frac{v-(t-1)}{k-(t-1)}\big) \\ \Longleftrightarrow \quad & t\text{-}(v,k,\lambda) &=& \ \mathrm{red}^{-1}\big((t-1)\text{-}\big(v,k,\lambda\frac{v-(t-1)}{k-(t-1)}\big)\big), \\ & \mathrm{der}\big(t\text{-}(v,k,\lambda)\big) &=& \quad (t-1)\text{-}(v-1,k-1,\lambda) \\ \Longleftrightarrow \quad & t\text{-}(v,k,\lambda) &=& \ \mathrm{der}^{-1}\big((t-1)\text{-}(v-1,k-1,\lambda)\big), \end{aligned}$$

and, if $k < v$,

$$\begin{aligned} & \mathrm{res}\big(t\text{-}(v,k,\lambda)\big) &=& \quad (t-1)\text{-}\big(v-1,k,\lambda\frac{v-k}{k-(t-1)}\big) \\ \Longleftrightarrow \quad & t\text{-}(v,k,\lambda) &=& \ \mathrm{res}^{-1}\big((t-1)\text{-}\big(v-1,k,\lambda\frac{v-k}{k-(t-1)}\big)\big). \end{aligned}$$

Rewriting the functions $\mathrm{red}^{-1}$, $\mathrm{der}^{-1}$ and $\mathrm{res}^{-1}$ in terms of arbitrary admissible $t$-$(v,k,\lambda)$ leads to the following definition. This is a partial definition in the sense that the functions are only defined for some admissible parameter sets $t$-$(v,k,\lambda)$:

**Definition 2.6** Let $t$-$(v,k,\lambda)$ be an admissible parameter set. Put (cf. Fig. 1)

(i) $\mathrm{red}^{-1} t\text{-}(v,k,\lambda) := (t+1)\text{-}(v,k,\lambda\frac{k-t}{v-t})$ if admissible

(ii) $\mathrm{der}^{-1} t\text{-}(v,k,\lambda) := (t+1)\text{-}(v+1,k+1,\lambda)$ if admissible and

(iii) $\mathrm{res}^{-1} t\text{-}(v,k,\lambda) := (t+1)\text{-}(v+1,k,\lambda\frac{k-t}{v+1-k})$ if admissible.

We must carefully check that the operators are defined on $t$-$(v,k,\lambda)$. In case they are, we call these the *inverse operators* of red, der and res, respectively, as we then have $\mathrm{red}\,\mathrm{red}^{-1} = \mathrm{red}^{-1}\,\mathrm{red} = \mathrm{id}$, $\mathrm{der}\,\mathrm{der}^{-1} = \mathrm{der}^{-1}\,\mathrm{der} = \mathrm{id}$, $\mathrm{res}\,\mathrm{res}^{-1} = \mathrm{res}^{-1}\,\mathrm{res} = \mathrm{id}$. Moreover, the inverse operators (and the

$$
\begin{tikzcd}
(t+1)\text{-}(v+1,k,\lambda\frac{k-t}{v+1-k}) & (t+1)\text{-}(v+1,k+1,\lambda) & (t+1)\text{-}(v,k,\lambda\frac{k-t}{v-t}) \\
& t\text{-}(v,k,\lambda) \arrow[ul, "\mathrm{res}^{-1}"] \arrow[u, "\mathrm{der}^{-1}"'] \arrow[ur, "\mathrm{red}^{-1}"'] \arrow[dl, "\mathrm{red}"'] \arrow[d, "\mathrm{der}"] \arrow[dr, "\mathrm{res}"] & \\
(t-1)\text{-}(v,k,\lambda\frac{v-(t-1)}{k-(t-1)}) & (t-1)\text{-}(v-1,k-1,\lambda) & (t-1)\text{-}(v-1,k,\lambda\frac{v-k}{k-(t-1)})
\end{tikzcd}
$$

Figure 1: The operators $\mathrm{red}^{\pm 1}$, $\mathrm{der}^{\pm 1}$ and $\mathrm{res}^{\pm 1}$

original ones) commute. If one of the operators is defined for a parameter set, we say that the parameters *extend* or *lift* under that operator. The next result tells us that the order in which we apply the operators to a parameter set is irrelevant.

**Lemma 2.7** *Let $t$-$(v,k,\lambda)$ be admissible, and assume that $D_2D_1\big(t\text{-}(v,k,\lambda)\big)$ is defined for $D_1, D_2 \in \{\mathrm{red}^{-1}, \mathrm{der}^{-1}, \mathrm{res}^{-1}\}$. Then $D_1^{-1}D_2D_1\big(t\text{-}(v,k,\lambda)\big) = D_2\big(t\text{-}(v,k,\lambda)\big)$ is admissible, too. In particular, if $\mathrm{red}^{-h}\mathrm{der}^{-i}\mathrm{res}^{-j}\big(t\text{-}(v,k,\lambda)\big)$ is admissible for some integers $h$, $i$ and $j$, then $\mathrm{red}^{-h'}\mathrm{der}^{-i'}\mathrm{res}^{-j'}\big(t\text{-}(v,k,\lambda)\big)$ is admissible for all $h' \le h$, $i' \le i$ and $j' \le j$.*

The operators are linear in the index in the following sense:

**Lemma 2.8** *Let $t$-$(v,k,\lambda)$ be an admissible design parameter set. Assume that for some nonnegative integers $h$, $i$ and $j$, $\mathrm{red}^{-h}\,\mathrm{der}^{-i}\,\mathrm{res}^{-j}\,t\text{-}(v,k,\lambda)$ is defined. Then for $m \le \lfloor \frac{\lambda_{\max}(t,v,k)}{\lambda} \rfloor$,*

$$\mathrm{red}^{\pm h}\,\mathrm{der}^{\pm i}\,\mathrm{res}^{\pm j}\,\big(t\text{-}(v,k,m\lambda)\big) = m\Big(\mathrm{red}^{\pm h}\,\mathrm{der}^{\pm i}\,\mathrm{res}^{\pm j}\,t\text{-}(v,k,\lambda)\Big),$$

*where $m\big(t'\text{-}(v',k',\lambda')\big) := t'\text{-}(v',k',m\lambda')$.*

*Proof:* Each of the operators $\mathrm{red}^{\pm 1}$, $\mathrm{der}^{\pm 1}$ and $\mathrm{res}^{\pm 1}$ is linear with respect to $\lambda$. □

We extend the result of Lemma 2.3

**Lemma 2.9** *Let $D\big(t\text{-}(v,k,\lambda)\big) = t'\text{-}(v',k',\lambda')$ be admissible for some $D \in \{\mathrm{red}^{\pm 1}, \mathrm{der}^{\pm 1}, \mathrm{res}^{\pm 1}\}$. and admissible $t$-$(v,k,\lambda)$. Then*

$$\frac{\lambda}{\lambda_{\max}(t,v,k)} = \frac{\lambda'}{\lambda_{\max}(t',v',k')}.$$

*Hence, the function $\lambda_{\max}$ is proportional to the index under the six operations. In particular, $D\big(t\text{-}(v,k,\lambda)\big)$ is complete if and only if $t$-$(v,k,\lambda)$ is complete.*

*Proof:* Half of the work has already been done in Lemma 2.3. Now

$$D\big(t\text{-}(v,k,\lambda)\big) = t'\text{-}(v',k',\lambda') \iff t\text{-}(v,k,\lambda) = D^{-1}\big(t'\text{-}(v',k',\lambda')\big)$$

implies that the result is true for $D^{-1}$, too. $\square$

# 3 Some constructions

Several constructions of $t$-designs from other $t$-designs are known. Three have already been described, namely the reduced design, the derived design and the residual design. There are more:

For a set of blocks $\mathcal{B}$ being subsets of a set $\mathcal{V}$ put $\overline{\mathcal{B}} = \{\mathcal{V} \setminus B \mid B \in \mathcal{B}\}$, the *complementary block system.* It is well known that the complementary block system also forms a $t$-design, called the *complementary design.* Two designs are isomorphic if and only if the corresponding complementary designs are isomorphic. For a set of blocks $\mathcal{B}$ and a further point $x$ we put $\mathcal{B} * \{x\} := \{B \cup \{x\} \mid B \in \mathcal{B}\}$.

**Theorem 3.1** *(Alltop [1]) Let $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ be a $t$-$(2k+1,k,\lambda)$ design with $\mathcal{V} = \{1,\ldots,2k+1\}$.*

*(i) If $t$ is even, then $\Big(\mathcal{V} \cup \{2k+2\}, \mathcal{B} * \{2k+2\} \cup \overline{\mathcal{B}}\Big)$ is a $(t+1)$-$(2k+2,k+1,\lambda)$ design.*

*(ii) If $t$ is odd and $\lambda = \frac{\lambda_{\max}}{2}$, then $\Big(\mathcal{V} \cup \{2k+2\}, \mathcal{B} * \{2k+2\} \cup \binom{\mathcal{V}}{k} \setminus \overline{\mathcal{B}}\Big)$ is a $(t+1)$-$(2k+2,k+1,\lambda)$ design.*

**Theorem 3.2** *(Tran van Trung [11], van Leijenhorst [8]) If for some parameters $t$-$(v,k,\lambda)$, designs $(\mathcal{V} \setminus \{v\}, \mathcal{B}_1)$ and $(\mathcal{V}, \mathcal{B}_2)$ with the parameters* $\operatorname{red} t$-$(v,k,\lambda)$ *and* $\operatorname{res} t$-$(v,k,\lambda)$ *exist, then $(\mathcal{V}, \mathcal{B}_1 * \{v\} \cup \mathcal{B}_2)$ is a design with parameters* $\operatorname{red} t$-$(v,k,\lambda)$. *In other words, if there exist $(t-1)$-$(v-1,k-1,\lambda)$ and $(t-1)$-$\big(v-1,k,\lambda\frac{v-k}{k-(t-1)}\big)$ designs, then there exists a $(t-1)$-$\big(v,k,\lambda\frac{v-(t-1)}{k-(t-1)}\big)$ design.*

Tran van Trung also noticed the special case $v = 2k+1$:

**Theorem 3.3** *(Tran van Trung) If there exists a $t$-$(2k+1,k,\lambda)$ design $(\mathcal{V},\mathcal{B})$ then $(\mathcal{V}, \mathcal{B} \cup \overline{\mathcal{B}})$ is a $t$-$(2k+2,k+1,\lambda\frac{2k+2-t}{k+1-t})$ design.*

We add a similar observation here derived from Theorem 3.1.

**Theorem 3.4** *Let $t$ be odd. If there exists a $t$-$(2k+2,k,\lambda)$ design with $k > t$, then there also exists a $t$-$\big(2k+2,k+1,\lambda\frac{k+2}{k-t+1}\big)$ design and a $t$-$\big(2k+3,k+1,\lambda\frac{2k+3-t}{k+1-t}\big)$ design and a $t$-$\big(2k+4,k+1,\lambda\frac{2k+3-t}{k+1-t}\frac{2k+4-t}{k+2-t}\big)$ design.*

*Proof:* Applying Theorem 3.1, (i) to the residual of the given design yields the $t$-$\left(2k+2, k+1, \lambda\frac{k+2}{k-t+1}\right)$ design. Combining this design with the original one, Theorem 3.2 yields a $t$-$\left(2k+3, k+1, \lambda\frac{2k+3-t}{k+1-t}\right)$. By Theorem 3.3, a $t$-$\left(2k+4, k+1, \lambda\frac{2k+3-t}{k+1-t}\frac{2k+4-t}{k+2-t}\right)$ design exists. $\square$

Observe that if in Theorem 3.4 the original design has automorphism group $A$, then all the other designs admit a point stabilizer in $A$ as a group of automorphisms, because we took the residual of the original design. More precisely, we have the stabilizer of the point we removed from $t-(2k+2, k, \lambda)$. In Alltop's construction all automorphisms of the given design extend to automorphisms of the constructed design, fixing the newly added point. These again extend to automorphisms of the design formed in the third step of construction.

## 4 Families of design parameter sets

**Definition 4.1** Let $t$-$(v, k, \lambda)$ be admissible. We define the *family of design parameters generated by* $t$-$(v, k, \lambda)$ as

$$\mathrm{Family}\big(t\text{-}(v,k,\lambda)\big) = \left\{\mathrm{red}^h \mathrm{der}^i \mathrm{res}^j t\text{-}(v,k,\lambda) \;\middle|\; h,i,j \in \mathbb{N},\ h+i+j \le t,\ j \le v-k\right\}.$$

By Lemma 2.4, all parameter sets of a family are admissible. We can draw the family as a tetrahedron or three-sided pyramid, with the generating parameter set at the top and the three directions pointing downwards standing for the three kinds of operators der, res and red. The *layers* of a family are the horizontal plane intersections. They contain the parameter sets with a fixed $t$. The pyramid of a $t$-$(v, k, \lambda)$ has $t+1$ layers, corresponding to $t$-, $(t-1)$-, ..., 1- and 0-design parameters, respectively. Figure 2 sketches the family of a parameter set $t$-$(v, k, \lambda)$.

We are going to describe the parameter sets of a family more directly. Let $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ be a $t$-$(v, k, \lambda)$ design. For $i + j \le t$ with $j \le v - k$ let $I$ and $J$ be disjoint subsets of $\mathcal{V}$ of size $i$ and $j$, respectively. Then the numbers

$$\lambda_{i,j} = \big|\{B \in \mathcal{B} \mid I \subseteq B \subseteq \mathcal{V} \setminus J\}\big|$$

do not depend on the choice of $I$ and $J$. They are usually called *generalized $\lambda$-values* of the design. They can be computed from the parameters of the design. For example, Ray-Chaudhuri and Wilson [10] proved that

$$\lambda_{i,j} = \lambda \frac{\binom{v-i-j}{k-i}}{\binom{v-t}{k-t}} \quad \text{for } i+j \le t.$$

Note that $\lambda = \lambda_{t,0}$ and $\lambda_{\mathrm{res}^i} = \lambda_{t-i,0}$ for $0 \le i \le t$ and that the previous formulae extend to formal parameter sets.

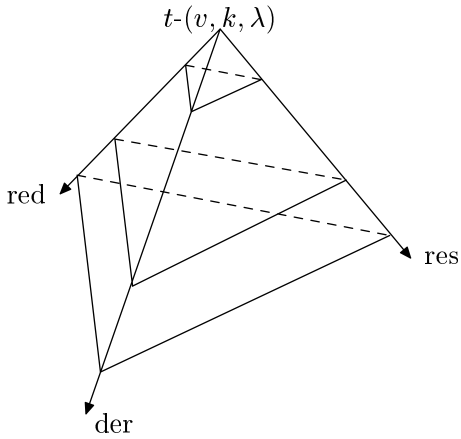



Figure 2: The family of a $t$-$(v,k,\lambda)$

Neglecting the question of existence of designs, we again switch over to consider only parameter sets of designs. We require the parameters to be admissible.

**Lemma 4.2** *Let $t$-$(v,k,\lambda)$ be admissible. Then, for nonnegative integers $h$, $i$ and $j$ satisfying $h+i+j \le t$ and $j \le v-k$,*

$$\mathrm{red}^h \mathrm{der}^i \mathrm{res}^j \, t\text{-}(v,k,\lambda) = (t-i-j-h)\text{-}\big(v-i-j, k-i, \lambda_{t-h-j,j}\big).$$

*Proof:* Let $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ be a putative $t$-$(v,k,\lambda)$ design. We will reduce the index of this design $h$ times, form the derived design $i$ times and take the residual design $j$ times. We only consider the parameters of the resulting design, making no use of the existence of $\mathcal{D}$ (which cannot be guaranteed).

Let $I$ and $J$ be disjoint subsets of $\mathcal{V}$ of size $i$ and $j$, respectively. We show that deriving at all points of $I$ and forming the residual with respect to all points of $J$ and reducing the index $h$ times yields a design $\mathcal{D}' = (\mathcal{V}', \mathcal{B}')$ having the stated parameters. The point set $\mathcal{V}' = \mathcal{V} \setminus (I \cup J)$ has size $v-i-j$ and the blocks are of size $k-i$. By the second assumption, $v-i-j \ge v-i-(v-k) = k-i$, i. e. the block size is less than or equal to the number of points in $\mathcal{D}'$. In order to determine the index, let $S$ be a subset of cardinality $t-i-j-h$ of $\mathcal{V}'$. The blocks of $\mathcal{B}'$ are in one-to-one correspondence with the blocks of $\mathcal{B}$ containing $I$ and avoiding $J$. Consider $W = S \cup I$ as a subset of $\mathcal{V}$ of size $t-j-h$. Since $W \cap J = \emptyset$, there are exactly $\lambda_{t-j-h,j}$ blocks of $\mathcal{D}$ containing $W$ and avoiding $J$. Exactly these blocks correspond to the blocks in $\mathcal{B}'$ containing $S$. $\square$

Note that by Lemma 2.4, the parameters in Lemma 4.2 are admissible.

**Theorem 4.3** *The parameter sets $t'$-$(v', k', \lambda')$ in the family generated by $t$-$(v, k, \lambda)$ are characterized by the following conditions:*

*(i)* $0 \le t' \le t$,

*(ii)* $k' \le v' \le v$,

*(iii)* $t' \le k' \le k$,

*(iv)* $k - k' \le v - v' \le t - t'$,

*(v)* $\lambda' = \lambda_{i,j}$, *where* $i = (k - k') + t'$ *and* $j = (v - v') - (k - k')$.

*Proof:* Let $t'$-$(v', k', \lambda')$ be a parameter set satisfying (i)-(v). Then $i := k-k'$ and $j := v-v'-(k-k')$ and $h := t - t' - i - j = t - t' - (v - v')$ are nonnegative integers with $h + i + j = t - t' \le t$, and $j \le v - k - (v' - k') \le v - k$. By Lemma 4.2,

$$\begin{aligned}\mathrm{red}^h \mathrm{der}^i \mathrm{res}^j\, t\text{-}(v, k, \lambda) &= (t - i - j - h)\text{-}\big(v - i - j, k - i, \lambda_{t-h-j,j}\big)\\ &= t'\text{-}\big(v', k', \lambda_{t'+(k-k'),v-v'-(k-k')}\big)\end{aligned}$$

and thus $t'$-$(v', k', \lambda') \in \mathrm{Family}\big(t\text{-}(v, k, \lambda)\big)$. A routine check using Lemma 4.2 shows that all design parameter sets contained in the family of a $t$-$(v, k, \lambda)$ satisfy (i)-(v). □

In Section 6, we will illustrate this result by the family of designs generated by 15-$(32, 16, 6)$.

## 5 Ancestors

For admissible $t$-$(v, k, \lambda)$, what are the families containing it? In case that there are more than one, what is the relationship between any two of them? We introduce the concept of an *ancestor parameter set* which helps answering these questions:

**Theorem 5.1** *Let $t$-$(v, k, \lambda)$ be an admissible parameter set with $\lambda \ne \binom{v-t}{k-t}$. Then there exists a unique largest admissible parameter set, called* Ancestor$\big(t$-$(v, k, \lambda)\big)$*, such that $t$-$(v, k, \lambda)$ is contained in its family. More precisely, there exist integers $h_{\max}$, $i_{\max}$ and $j_{\max}$ maximal with respect to the property that*

$$\mathrm{Ancestor}\big(t\text{-}(v, k, \lambda)\big) = \mathrm{red}^{-h_{\max}}\, \mathrm{der}^{-i_{\max}}\, \mathrm{res}^{-j_{\max}}\, t\text{-}(v, k, \lambda)$$

*is defined. The given parameter set is the $h_{\max}$-fold reduction, $i_{\max}$-fold derived and $j_{\max}$-fold residual of its ancestor. A design parameter set which equals its own ancestor is called* ancestor parameter set. *The families containing $t$-$(v, k, \lambda)$ are generated by parameter sets belonging to* *the family of the ancestor. In particular, they are all contained in the family of the ancestor. If* $\mathrm{Ancestor}\big(t\text{-}(v,k,\lambda)\big) = t'\text{-}(v',k',\lambda')$ *then*

$$\frac{\lambda'}{\lambda_{\max}(t',v',k')} = \frac{\lambda}{\lambda_{\max}(t,v,k)}.$$

*In particular, if the parameter set is not complete, the ancestor is not complete as well.*

Before we start proving the result, we add some preliminary results about admissibility of parameter sets:

**Lemma 5.2** *Let* $t\text{-}(v,k,\lambda)$ *be admissible, and assume that* $D_1\big(t\text{-}(v,k,\lambda)\big)$ *and* $D_2\big(t\text{-}(v,k,\lambda)\big)$ *are definined for* $D_1, D_2 \in \{\mathrm{red}^{-1}, \mathrm{der}^{-1}, \mathrm{res}^{-1}\}$, $D_1 \neq D_2$*. In addition, if* $\{D_1, D_2\} = \{\mathrm{red}^{-1}, \mathrm{res}^{-1}\}$ *we assume that* $\lambda \neq \lambda_{\max}$*. Then* $D_1 D_2\big(t\text{-}(v,k,\lambda)\big) = D_2 D_1\big(t\text{-}(v,k,\lambda)\big)$ *is admissible, too.*

*Proof:* We distinguish 3 cases according to $D_1, D_2$. Up to a reordering of $D_1$ and $D_2$ these are all possible cases.

$D_1 = \mathrm{der}^{-1}, D_2 = \mathrm{res}^{-1}$ : We do not yet know if $\mathrm{der}^{-1}\mathrm{res}^{-1}\big(t\text{-}(v,k,\lambda)\big)$ is admissible, but the commutativity of the operators allows to deduce

$$\mathrm{res}\big(\mathrm{der}^{-1}\mathrm{res}^{-1} t\text{-}(v,k,\lambda)\big) = \mathrm{der}^{-1} t\text{-}(v,k,\lambda)$$

and

$$\mathrm{der}\big(\mathrm{der}^{-1}\mathrm{res}^{-1} t\text{-}(v,k,\lambda)\big) = \mathrm{res}^{-1} t\text{-}(v,k,\lambda)$$

are admissible by assumption. Hence by Proposition 2.5, (iv) $\Leftrightarrow$ (i), $\mathrm{der}^{-1}\mathrm{res}^{-1} t\text{-}(v,k,\lambda)$ is admissible, i. e. $D_1 D_2\big(t\text{-}(v,k,\lambda)\big)$ is defined.

$D_1 = \mathrm{red}^{-1}, D_2 = \mathrm{der}^{-1}$ : We can proceed in a similar way using Proposition 2.5, (v) $\Leftrightarrow$ (i), respectively, to get the result in that case.

$D_1 = \mathrm{red}^{-1}, D_2 = \mathrm{res}^{-1}$ : In this case we have the additional assumption $\lambda \neq \lambda_{\max}$ which we need to show that $\mathrm{red}^{-1}\mathrm{res}^{-1}\big(t\text{-}(v,k,\lambda)\big)$ satisfies (ADM1): Deny this. Then $t+1 = k$ and $\mathrm{red}^{-1} t\text{-}(v,t+1,\lambda) = (t+1)\text{-}\big(v,t+1,\lambda\frac{1}{v-t}\big)$ and $\mathrm{res}^{-1} t\text{-}(v,t+1,\lambda) = (t+1)\text{-}\big(v+1,t+1,\lambda\frac{1}{v-t}\big)$. But the last two parameter sets are complete, hence $t\text{-}(v,k,\lambda)$ is complete by Lemma 2.9, contradicting the assumption $\lambda \neq \lambda_{\max}$. We conclude that $t \leq k-2$, so $\mathrm{red}^{-1}\mathrm{res}^{-1}\big(t\text{-}(v,k,\lambda)\big)$ satisfies (ADM1). We proceed as usual:

$$\mathrm{res}\big(\mathrm{red}^{-1}\mathrm{res}^{-1} t\text{-}(v,k,\lambda)\big) = \mathrm{red}^{-1} t\text{-}(v,k,\lambda)$$

and

$$\mathrm{red}\big(\mathrm{red}^{-1}\mathrm{res}^{-1}t\text{-}(v,k,\lambda)\big) = \mathrm{res}^{-1}t\text{-}(v,k,\lambda)$$

are admissible by assumption. Moreover, $t+1 < k$, which means that the two parameter sets on the right hand side satisfy the additional assumption of Proposition 2.5, (vi). Hence $\mathrm{red}^{-1}\mathrm{res}^{-1}t\text{-}(v,k,\lambda)$ is admissible, i. e. $D_1D_2\big(t\text{-}(v,k,\lambda)\big)$ is defined.

□

**Lemma 5.3** *Let $t\text{-}(v,k,\lambda)$ be an admissible design parameter set with $\lambda \neq \binom{v-t}{k-t}$. Assume that for nonnegative integers $h_1$, $h_2$, $i_1$, $i_2$, $j_1$, $j_2$ the parameter sets*

$$\mathrm{red}^{-h_\ell}\,\mathrm{der}^{-i_\ell}\,\mathrm{res}^{-j_\ell}\,t\text{-}(v,k,\lambda) \quad \textit{are admissible for} \quad \ell = 1,2. \tag{6}$$

*Then $\mathrm{red}^{-\max(h_1,h_2)}\,\mathrm{der}^{-\max(i_1,i_2)}\,\mathrm{res}^{-\max(j_1,j_2)}\,t\text{-}(v,k,\lambda)$ is admissible as well.*

*Proof:* First note that $\mathrm{red}^{-\min(h_1,h_2)}\,\mathrm{der}^{-\min(i_1,i_2)}\,\mathrm{res}^{-\min(j_1,j_2)}\,t\text{-}(v,k,\lambda)$ is admissible by Lemma 2.7. Hence it suffices to prove the claim for the case that one of the corresponding integers is zero, i. e. $h_1h_2 = 0$, $i_1i_2 = 0$ and $j_1j_2 = 0$. The assumption is still given by (6). Assume that $h_2 \neq 0$, hence $h_1 = 0$. In this case we pick $\mathrm{red}^{-1}$ and use Lemma 5.2 to show that $\mathrm{red}^{-1}\mathrm{der}^{-i_1}\mathrm{res}^{-j_1}\,t\text{-}(v,k,\lambda)$ is admissible. By Lemma 2.7, $\mathrm{der}^{-i_1}\mathrm{res}^{-j_1}\big(\mathrm{red}^{-1}t\text{-}(v,k,\lambda)\big)$ is admissible, as well as $\mathrm{der}^{-i_2}\mathrm{res}^{-j_2}\big(\mathrm{red}^{-1}t\text{-}(v,k,\lambda)\big)$, i. e. the assumption (6) holds for $\mathrm{red}^{-1}t\text{-}(v,k,\lambda)$ instead of $t\text{-}(v,k,\lambda)$ and $h_2$ reduced by one. We proceed by induction. Similarly, we proceed with the other operators. The assumption $\lambda \neq \binom{v-t}{k-t}$ is needed for applying Lemma 5.2. □

*Proof:* (of Theorem 5.1) Fix an admissible parameter set $t\text{-}(v,k,\lambda)$ with $\lambda < \binom{v-t}{k-t} = \lambda_{\max}$. We first have to show that $\mathrm{Ancestor}\big(t\text{-}(v,k,\lambda)\big)$ is defined, i. e. that the integers $h_{\max}$, $i_{\max}$ and $j_{\max}$ exist. Therefore, we look at the parameters arising as results of the operations $\mathrm{der}^{-1}$, $\mathrm{red}^{-1}$ and $\mathrm{res}^{-1}$ (cf. Fig. 1). In case of $\mathrm{red}^{-1}$ and $\mathrm{res}^{-1}$, the difference $k-t$ strictly decreases, so $i_{\max}$ and $j_{\max}$ are both bounded above by $k-t$. What can be said about the number of times that $\mathrm{red}^{-1}$ can be applied? Assume this is the case infinitly often and put

$$b_n = \#\text{ of blocks of } \mathrm{der}^{-n}\,t\text{-}(v,k,\lambda) = \lambda\frac{\binom{v+n}{t+n}}{\binom{k+n}{t+n}}.$$

By (ADM3-0), $b_n$ is an integer. Note that

$$\frac{b_{n+1}}{b_n} = \frac{v+n+1}{k+n+1} \tag{7}$$

holds, even for the complete design with $\lambda = \lambda_{\max}(t, v, k)$. For simplicity, we write $\lambda_{\max}$ instead of $\lambda_{\max}(t, v, k)$. Put

$$c_n = \# \text{ of blocks of } \mathrm{der}^{-n}\, t\text{-}(v, k, \lambda_{\max}) = \lambda_{\max} \frac{\binom{v+n}{t+n}}{\binom{k+n}{t+n}} = \binom{v+n}{k+n}.$$

By (7), the sequences of numbers $(b_n)$ and $(c_n)$ are proportional. Hence there exists a rational number $\gamma$ with $0 < \gamma < 1$ and $\gamma = \frac{b_n}{c_n}$ for all $n \geq 0$. In particular,

$$\gamma = \frac{b_0}{c_0} = \frac{\binom{v}{t}\lambda}{\binom{k}{t}\binom{v}{k}} = \frac{\binom{v}{t}\lambda}{\binom{v}{t}\binom{v-t}{k-t}} = \frac{\lambda}{\lambda_{\max}}.$$

Moreover,

$$b_n = \gamma c_n = \frac{\lambda \binom{v+n}{k+n}}{\lambda_{\max}} \in \mathbb{N}$$

for all integers $n \geq 0$. Let $p$ be a prime dividing $\frac{\lambda_{\max}}{\gcd(\lambda_{\max}, \lambda)}$ (such a prime exists since $0 < \lambda < \lambda_{\max}$ by assumption). We deduce that $p$ divides $\binom{v+n}{k+n}$ for all $n$. But this is impossible as for example $\binom{p^m-1}{j} = \prod_{h=1}^{j} \frac{p^m-h}{h} \not\equiv 0 \mod p$ for all $0 \leq j \leq p^m - 1$ and arbitrary $m$ : If the numerator is divisible by $p^s$, say, then the denominator is divisible by that number, too. Hence all factors $p$ cancel in the product. We conclude that the number of times that $\mathrm{red}^{-1}$ can be applied is finite, and we let $h_{\max}$ be maximal with respect to the property that $\mathrm{res}^{-h_{\max}}$ is defined.

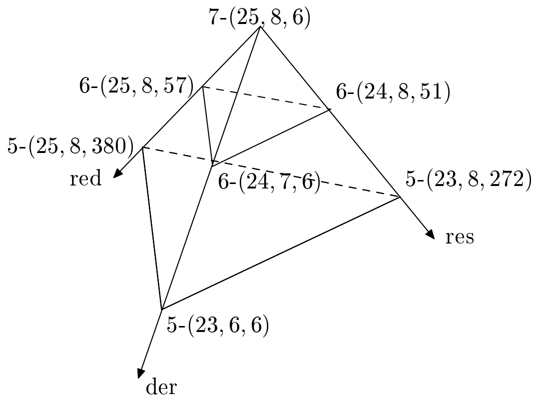
Lemma 5.3 implies the uniqueness of the ancestor parameter set. The remaining statement follows by repeated application of Lemma 2.9. □

**Example 5.4** There exist 5-(23, 6, 6) designs invariant under $\mathrm{Hol}(\mathrm{C}_{23})$. We find that

$$\mathrm{der}^{-2}\, 5\text{-}(23, 6, 6) = 7\text{-}(25, 8, 6)$$

is ancestor. (Note that it is not known whether or not a 7-(25, 8, 6) design exists.) Figure 3 displays the family of this parameter set. On the left, the pyramid of parameter sets with $t \geq 5$ is shown Note that in the bottom layer, there are three more design parameter sets. A more concise way of displaying the family is indicated to the right, which shows the layers one after another. The underlined parameter sets are known to be realizable. ◇

The ancestors of designs differing only by a multiple of $\Delta\lambda$ may look very different:




7-(25, 8, 6)

---

6-(25, 8, 57)   6-(24, 8, 51)
6-(24, 7, 6)

---

$\underline{\text{5-(25,8,380)}}$   $\underline{\text{5-(24,8,323)}}$   5-(23, 8, 272)
$\underline{\text{5-(24,7,57)}}$   5-(23, 7, 51)
$\underline{\text{5-(23,6,6)}}$

Figure 3: The family of 7-(25, 8, 6)

**Example 5.5**

$$\text{Ancestor}\big(5\text{-}(24,8,\lambda)\big) = \begin{cases} 17\text{-}(36,18,m\cdot 1) & \text{if } \lambda = m\cdot 51 \\ 15\text{-}(32,16,m\cdot 1) & \text{if } \lambda = m\cdot 57 \\ 13\text{-}(32,16,m\cdot 3) & \text{if } \lambda = m\cdot 3,\ 51\nmid\lambda,\ 57\nmid\lambda \\ 6\text{-}(25,8,m\cdot 3) & \text{if } \lambda = m\cdot 17, 19\nmid\lambda \\ 6\text{-}(24,8,m\cdot 3) & \text{if } \lambda = m\cdot 19, 17\nmid\lambda \\ 7\text{-}(25,8,m\cdot 6) & \text{if } \lambda = m\cdot 323 = 17\cdot 19 \\ 5\text{-}(24,8,\lambda) & \text{else} \end{cases}$$

Note that the "else" case is equivalent to

$$\gcd\left(\lambda, \frac{\lambda_{\max}(5,24,8)}{\Delta\lambda(5,24,8)}\right) = \gcd(\lambda, 969/1) = \gcd(\lambda, 3\cdot 17\cdot 19) = 1.$$

◇

In a subsequent paper these ancestor relations will be examined more precisely. In a special case we can evaluate the ancestor theoretically. We remark that C. Colbourn and R. Mathon in [4] already tabulated the largest $t$ for which a $t$-$(v, t+1, 1)$ design is possible.

**Proposition 5.6** *Let $t$ and $v$ be integers with $v - t$ a prime. Then for natural $n$ :*

*(i)* $(t+n)$-$\big(v+n, t+n+1, 1\big)$ *is admissible* $\iff n \le v - 2(t+1)$;

*(ii)* $\text{Ancestor}(t\text{-}(v, t{+}1, 1)) = \text{der}^{-n}\, t\text{-}(v, t{+}1, 1)$ *with* $n = v - 2(t+1)$. *In particular, if* $\text{Ancestor}(t\text{-}(v, t+1, 1)) = t'\text{-}(v', k', 1)$ *then* $\frac{k'}{v'} = \frac{1}{2}$.

*Proof:*

(i) We have the following equivalence:

$$\begin{aligned} & (t+n)\text{-}\big(v+n, t+n+1, 1\big) \text{ is admissible} \\ \Longleftrightarrow\ & (\text{ADM3-}s) \text{ holds for } 0 \le s \le t+n \\ \Longleftrightarrow\ & (\text{ADM3-}(t+n-s)) \text{ holds for } 0 \le s \le t+n \\ \Longleftrightarrow\ & \frac{(v-(t-1))\cdots(v-(t-s))}{2\cdots(s+1)} = \frac{1}{v-t}\binom{v+t+s}{s+1} \text{ is integral for } 0 \le s \le t+n. \end{aligned}$$

If $s+1 < v-t$ then $v-t$ prime implies $\gcd\big((s+1)!, v-t\big) = 1$. Since $\binom{v+t+s}{s+1}$ is an integer for all $s$,

$$(s+1)! \;\Big|\; (v-(t-1))\cdots(v-(t-s))$$

for $s+1 < v-t \Leftrightarrow s \le v-t-2$. Hence $(t+n)\text{-}\big(v+n, t+n+1, 1\big)$ is admissible for all $n$ satisfying $t+n \le v-t-2 \Leftrightarrow n \le v-2(t-1)$. On the other hand, if $n = v-2t-1$ then the parameter set in question would be $(v-t-1)\text{-}(2v-2t-1, v-t, 1)$, so $\big(\text{ADM3-}0\big)$ would require that

$$\frac{(2v-2t-1-(v-t-2))\cdots(2v-2t-1)}{(v-t-(v-t-2))\cdots(v-t-0)} = \frac{(v-t+1)\cdots(2(v-t)-1)}{2\cdots(v-t)}$$

were integral. But the prime $v-t$ divides none of the factors in the numerator, a contradiction.

(ii) Part (i) implies that $\text{der}^{-n}\, t-(v, t+1, 1) = (t+n)\text{-}\big(v+n, t+n+1, 1\big)$ is defined for $n = v-2(t+1)$ and no larger $n$. We claim that it is not possible to apply $\text{red}^{-1}$ or $\text{res}^{-1}$ to this parameter set: If $\text{red}^{-1}$ were applicable, then $(v+n)-(t+n-1) = v-t+1$ had to divide $1\cdot(t+n+1-(t+n)) = 1$ which it does not (since $v-t \ne 0$). If $\text{res}^{-1}$ were applicable, then $v+n+1-(t+n+1) = v-t$ had to divide $1$ which it does not (since $v-t \ne 1$). Hence, $\text{Ancestor}\big(t-(v, t+1, 1)\big) = \text{der}^{-n}\, t-(v, t+1, 1)$ with $n = v-2(t+1)$. In particular, if $\text{Ancestor}\big(t-(v, t+1, 1)\big) = t'-(v', k', 1) = (t+n)\text{-}\big(v+n, t+n+1, 1\big)$, then

$$\frac{k'}{v'} = \frac{t+n+1}{v+n} = \frac{v-t-1}{2v-2t-2} = \frac{1}{2}.$$

□

The design parameter sets involved can get quite large:

**Example 5.7**

$$\begin{aligned}
\text{Ancestor}\big(5\text{-}(244,6,1)\big) &= \text{der}^{-232}\,\big(5\text{-}(244,6,1)\big) = 237\text{-}(476,238,1),\\
\text{Ancestor}\big(5\text{-}(168,6,1)\big) &= \text{der}^{-156}\,\big(5\text{-}(168,6,1)\big) = 161\text{-}(324,162,1),\\
\text{Ancestor}\big(5\text{-}(132,6,1)\big) &= \text{der}^{-120}\,\big(5\text{-}(132,6,1)\big) = 125\text{-}(252,126,1),\\
\text{Ancestor}\big(5\text{-}(108,6,1)\big) &= \text{der}^{-96}\,\big(5\text{-}(108,6,1)\big) = 101\text{-}(204,102,1).
\end{aligned}$$

(The existence of the mentioned Steiner 5-systems has been proved by Grannell, Griggs, Mathon [5] as well as by Mathon [9] and by Betten, Molodtsov, Laue and Wassermann [2]. Note that for all known 5-$(v,6,1)$ designs $v-t$ is a prime.) $\diamond$

There are examples where the ancestor is a combination of all three operators:

**Example 5.8**

$$\begin{aligned}
\text{Ancestor}\big(7\text{-}(40,11,11220)\big) &= \text{red}^{-1}\text{der}^{-19}\text{res}^{-1}\,\big(7\text{-}(40,11,11220)\big) = 28\text{-}(60,30,136),\\
\text{Ancestor}\big(7\text{-}(40,11,8250)\big) &= \text{red}^{-1}\text{der}^{-4}\text{res}^{-1}\,\big(7\text{-}(40,11,8250)\big) = 13\text{-}(45,15,100),\\
\text{Ancestor}\big(6\text{-}(33,9,225)\big) &= \text{red}^{-1}\text{der}^{-3}\text{res}^{-1}\,\big(6\text{-}(33,9,225)\big) = 11\text{-}(37,12,2).
\end{aligned}$$

$\diamond$

## 6 Isomorphism types

In [7], the third author has related the isomorphism types of designs that can be constructed by the extension method of van Leijenhorst and Tran van Trung (Theorem 3.2) to double cosets in groups. The example of 7-$(26,10,342)$ designs presented there can be greatly extended within the corresponding family of designs. Let us start out by recalling a result of [7] and deriving another version of the bound of isomorphism types of designs.

**Theorem 6.1** *Let $\mathcal{D}_1 = (\mathcal{V},\mathcal{B}_1)$ be a $t$-$(v,k-1,\lambda)$ design with automorphism group $A_1$ and $\mathcal{D}_2 = (\mathcal{V},\mathcal{B}_2)$ be a $t$-$(v,k,\lambda(v-k+1)/(k-t))$ design with automorphism group $A_2$, defined on the same point set $\mathcal{V} = \{1,\cdots,v\}$. Put $\mathcal{V}' = \mathcal{V}\cup\{v+1\}$.*

*Let $\mathcal{D}(\pi_1) = (\mathcal{V}',\mathcal{B}_1 * \{v+1\}\cup\mathcal{B}_2^{\pi_1})$ and $\mathcal{D}(\pi_2) = (\mathcal{V}',\mathcal{B}_1 * \{v+1\}\cup\mathcal{B}_2^{\pi_2})$ be two design extensions and $\mathcal{D} = (\mathcal{V}',\mathcal{B})$ some $t$-design all with the parameters $t$-$(v+1,k,\lambda(v-t+1)/(k-t))$.*

*Suppose, there exist design isomorphisms $\phi_1,\phi_2 \in S_{\mathcal{V}'}$ with $\phi_1(\mathcal{D}) = \mathcal{D}(\pi_1)$ and $\phi_2(\mathcal{D}) = \mathcal{D}(\pi_2)$. Then there esists an isomorphism $\phi \in S_{\mathcal{V}'}$ with $\phi(\mathcal{D}(\pi_1)) = \mathcal{D}(\pi_2)$ and $\phi(x) = x$ for some $x \in \mathcal{V}'$ if and only if there exists an automorphism $\alpha$ of $\mathcal{D}$ mapping the point $\phi_1^{-1}(x)$ to the point $\phi_2^{-1}(x)$. In particular,*

$$Aut(\mathcal{D}(\pi_1))_{v+1} = A_1 \cap A_2^{\pi_1}.$$

*The number of isomorphism types of extensions is at least*

$$\frac{1}{v+1}|A_1 \backslash \mathcal{S}_\mathcal{V} / A_2|.$$

**Theorem 6.2** *If there are $n_1$ isomorphism types of $t$-$(v, k-1, \lambda)$ designs and $n_2$ isomorphism types of $t$-$(v, k, \lambda(v-k+1)/(k-t))$ designs with automorphism groups of order at most $a$ then there exist at least*

$$\frac{n_1 \cdot n_2 \cdot v!}{(v+1)a^2} \tag{8}$$

*isomorphism types of $t$-$(v+1, k, \lambda(v-t+1)/(k-t))$ designs with automorphism group of order at most $a(v+1)$.*

*Proof:* We assume that the given designs have the point set $\mathcal{V} = \{1, \cdots, v\}$. We first consider the new point $v+1$ as a distinguished point of these designs such that isomorphisms of these *rooted* designs have to fix this point. Then by the theorems 3.2 and 6.1, for each pair of $t$-$(v, k-1, \lambda)$ and $t$-$(v, k, \lambda(v-k+1)/(k-t))$ designs the number of classes of design extensions with distinguished point $v+1$ is $|A_1 \backslash \mathcal{S}_\mathcal{V} / A_2|$ where $A_1$, $A_2$ are the corresponding automorphism groups. The double cosets $A_1 \backslash \mathcal{S}_\mathcal{V} / A_2$ can be interpreted as orbits of $A_2$ on the set of right cosets of $A_1$ in $\mathcal{S}_\mathcal{V}$. The smallest number of double cosets is achieved if each orbit has maximal length, that is $|A_2|$. So, there are at least $\frac{v!}{|A_1| \cdot |A_2|}$ double cosets. The assumption $|A_1| \leq a$ and $|A_2| \leq a$ yields at least $\frac{v!}{a^2}$ classes of design extensions with distinguished point $v+1$ for each given pair of designs. Also, the group of automorphisms fixing $v+1$ is isomorphic to a subgroup of $A_1$ and $A_2$. Thus, its order is at most $a$.

For $n_1 \cdot n_2$ pairs there result at least $n_1 \cdot n_2 \frac{v!}{a^2}$ classes. Making $v+1$ an ordinary point fuses at most $v+1$ classes of rooted designs to ordinary isomorphism classes. The full automorphism group of a design then has the stabilizer of the point $v+1$ as a subgroup of index at most $v+1$. Therefore, there are at least

$$\frac{n_1 \cdot n_2 \cdot v!}{(v+1)a^2}$$

isomorphism types, each with an automorphism group of order at most $a(v+1)$. □

## 6.1 The family of designs with ancestor 15-$(32, 16, 6)$

We are picking up the estimation of the number of isomorphism types of designs presented in [7] for 7-$(26, 10, 342)$ designs. Note that this parameter set belongs to the family with the ancestor

15-$(32,16,6)$. In order to apply Theorem 3.2 most successful, it is required to have a lot of existence results for parameter sets of a base line.

We consider the layer of 7-designs in that family, and present the base lines for 7-, 8-, ... designs up to the ancestor parameter set 15-$(32,16,6)$ in Table 1. As we are primarily interested

15-(32,16,6)
14-(31,15,6) 14-(31,16,48)
13-(30,14,6) 13-(30,15,48) 13-(30,16,240)
12-(29,13,6) 12-(29,14,48) 12-(29,15,240) 12-(29,16,840)
11-(28,12,6) 11-(28,13,48) 11-(28,14,240) 11-(28,15,840) 11-(28,16,2184)
10-(27,11,6) 10-(27,12,48) 10-(27,13,240) 10-(27,14,840) 10-(27,15,2184) 10-(27,16,4368)
9-(26,10,6) 9-(26,11,48) 9-(26,12,240) 9-(26,13,840) 9-(26,14,2184) 9-(26,15,4368) 9-(26,16,6864)
8-(25,9,6)8-(25,10,48)8-(25,11,240)8-(25,12,840)8-(25,13,2184)8-(25,14,4368)8-(25,15,6864)8-(25,16,8580)
7-(24,8,6) 7-(,9,48) 7-(,10,240) 7-(,11,840) 7-(,12,2184) 7-(,13,4368) 7-(,14,6864) 7-(,15,8580) 7-(24,16,8580)

Table 1: Top of the pyramid for 15-(32,16,6)

in 7-designs, we need the $\lambda_{i,j}$ for $i \geq 7$, shown in Table 2. At the time of writing of [7], only the three leftmost designs in the baseline of 7-designs have been known to exist. Thus, designs with parameters 7-$(24,8,6)$ 7-$(24,9,48)$ and 7-$(24,10,240)$ have been known then.

Meanwhile, progress has been made with filling in more existence results of designs in that particular base line. More precisely, a large lower bound for the number of isomorphism types of 7-$(26,10,342)$, 7-$(25,9,54)$ and 7-$(25,10,288)$ designs has been derived from the number of isomorphism types of designs with parameters 7-$(24,8,6)$, 7-$(24,9,48)$, 7-$(24,10,240)$, whose automorphism group contains $\mathrm{PSL}(2,23)$. These parameter sets all belong to the family with ancestor 15-$(32,16,6)$.

Using refined searching techniques, we now were able to construct 7-$(24,11,840)$ designs with automorphism group $\mathrm{PGL}(2,23)$. Theorem 3.4 then yields 7-$(24,12,2184)$ designs, which can also be found directly starting with the group $\mathrm{PGL}(2,23)$. The complementary designs (in the sense of Section 3) have the parameters 7-$(24,k,\lambda)$ with $(k,\lambda) \in \{(13,4368), (14,6864), (15,8580), (16,17160)\}$. This means that the full base line of parameter sets of 7-designs in the family of 15-$(32,16,6)$ is known to be realizable. By Theorem 3.2, the whole layer of 7-designs can be realized. Notably, no 8-design of that family has yet been found.

We are now going to give an estimation of the number of isomorphism types of 7-designs in that family using Theorem 6.2. The number of isomorphism types of designs for parameter sets of the base line are determined using Corollary 2.3 from [3]:

| $\lambda_{i,j}$ | $j=0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $i=7$ | 721050 | 461472 | 288420 | 175560 | 103740 | 59280 | 32604 | 17160 | 8580 |
| 8 | 259578 | 173052 | 112860 | 71820 | 44460 | 26676 | 15444 | 8580 | |
| 9 | 86526 | 60192 | 41040 | 27360 | 17784 | 11232 | 6864 | | |
| 10 | 26334 | 19152 | 13680 | 9576 | 6552 | 4368 | | | |
| 11 | 7182 | 5472 | 4104 | 3024 | 2184 | | | | |
| 12 | 1710 | 1368 | 1080 | 840 | | | | | |
| 13 | 342 | 288 | 240 | | | | | | |
| 14 | 54 | 48 | | | | | | | |
| 15 | 6 | | | | | | | | |

Table 2: The values of $\lambda_{i,j}$ of the family with ancestor 15-$(32, 16, 6)$ for $i + j \leq t = 15$, $i \geq 7$

| parameter set | 7-(24,8,6) | 7-(24,9,48) | 7-(24,10,240) | 7-(24,11,840) | 7-(24,12,2184) |
|---|---|---|---|---|---|
| # of nonisom. designs | $\geq 132$ | $\geq 2827$ | $\geq 91$ | $\geq 155$ | $\geq 28500$ |

for the first five parameter sets. Switching over to the complementary designs, we get the existence results for the remaining designs:

| parameter set | 7-(24,13,4368) | 7-(24,14,6864) | 7-(24,15,8580) | 7-(24,16,8580) |
|---|---|---|---|---|
| # of nonisom. designs | $\geq 155$ | $\geq 91$ | $\geq 2827$ | $\geq 132$ |

Starting with $v = p+1$ and $a = |\mathrm{PGL}(2,p)| = (p+1)p(p-1) = \frac{(p+1)!}{(p-2)!}$, the number of isomorphism types of $(t+1)$-$(v+1, k, \lambda)$ designs is at least

$$n_1 \cdot n_2 \cdot \frac{v!}{(v+1)} \frac{(p-2)!}{v!} \frac{(p-2)!}{v!} = n_1 \cdot n_2 \cdot \frac{(p-2)!^2}{(v+1)!}$$

by (8). The order of the automorphism group of these designs is at most $\frac{(v+1)!}{(p-2)!}$. Repeating the extension process $i$ times lets the automorphism group grow up to at most order $\frac{(v+i)!}{(p-2)!}$. Figure 4 displays the bound for the number of nonisomorphic designs we get from Theorem 6.2. The leftmost $3 \times 3$ triangle is taken from [7].

# 7 Acknowledgement

The second author thanks the Deutsche Forschungsgemeinschaft for financial support by the project Ke 201/17-2.

$7-(32,16,\lambda_{7,0})$

$\geq 10^{4165}$

$7-(31,16,\lambda_{7,1})$

$\geq 10^{2080}$

$7-(30,16,\lambda_{7,2})$

$7-(31,15,\lambda_{8,0})$ $\quad \geq 10^{1035}$

$\geq 10^{2080}$ $\quad 7-(29,16,\lambda_{7,3})$

$7-(30,15,\lambda_{8,1})$ $\quad \geq 10^{511}$

$\geq 10^{1039}$ $\quad 7-(28,16,\lambda_{7,4})$

$7-(29,15,\lambda_{8,2})$ $\quad \geq 10^{252}$

$7-(30,14,\lambda_{9,0})$ $\quad \geq 10^{514}$ $\quad 7-(27,16,\lambda_{7,5})$

$\geq 10^{1035}$ $\quad 7-(28,15,\lambda_{8,3})$ $\quad \geq 10^{125}$

$7-(29,14,\lambda_{9,1})$ $\quad \geq 10^{250}$ $\quad 7-(26,16,\lambda_{7,6})$

$\geq 10^{514}$ $\quad 7-(27,15,\lambda_{8,4})$ $\quad \geq 10^{62}$

$7-(28,14,\lambda_{9,2})$ $\quad \geq 10^{117}$ $\quad 7-(25,16,\lambda_{7,7})$

$7-(29,13,\lambda_{10,0})$ $\quad \geq 10^{255}$ $\quad 7-(26,15,\lambda_{8,5})$ $\quad \geq 10^{20}$

$\geq 10^{511}$ $\quad 7-(27,14,\lambda_{9,3})$ $\quad \geq 10^{51}$ $\quad 7-(24,16,\lambda_{7,8})$

$7-(28,13,\lambda_{10,1})$ $\quad \geq 10^{123}$ $\quad 7-(25,15,\lambda_{8,6})$ $\quad \geq 132$

$\geq 10^{250}$ $\quad 7-(26,14,\lambda_{9,4})$ $\quad \geq 10^{19}$

$7-(27,13,\lambda_{10,2})$ $\quad \geq 10^{54}$ $\quad 7-(24,15,\lambda_{8,7})$

$7-(28,12,\lambda_{11,0})$ $\quad \geq 10^{123}$ $\quad 7-(25,14,\lambda_{9,5})$ $\quad \geq 2827$

$\geq 10^{252}$ $\quad 7-(26,13,\lambda_{10,3})$ $\quad \geq 10^{19}$

$7-(27,12,\lambda_{11,1})$ $\quad \geq 10^{57}$ $\quad 7-(24,14,\lambda_{9,6})$

$\geq 10^{117}$ $\quad 7-(25,13,\lambda_{10,4})$ $\quad \geq 91$

$7-(26,12,\lambda_{11,2})$ $\quad \geq 10^{22}$

$7-(27,11,\lambda_{12,0})$ $\quad \geq 10^{54}$ $\quad 7-(24,13,\lambda_{10,5})$

$\geq 10^{125}$ $\quad 7-(25,12,\lambda_{11,3})$ $\quad \geq 155$

$7-(26,11,\lambda_{12,1})$ $\quad \geq 10^{22}$

$\geq 10^{51}$ $\quad 7-(24,12,\lambda_{11,4})$

$7-(25,11,\lambda_{12,2})$ $\quad \geq 28500$

$7-(26,10,\lambda_{13,0})$ $\quad \geq 10^{19}$

$\geq 10^{62}$ $\quad 7-(24,11,\lambda_{12,3})$

$7-(25,10,\lambda_{13,1})$ $\quad \geq 155$

$\geq 10^{19}$

$7-(24,10,\lambda_{13,2})$

$7-(25,9,\lambda_{14,0})$ $\quad \geq 91$

$\geq 10^{20}$

$7-(24,9,\lambda_{14,1})$

$\geq 2827$

$7-(24,8,\lambda_{15,0})$

$\geq 132$

Figure 4: The layer of 7-designs with a lower bound on the number of isomorphism types below each parameter set

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