# Isomorphism Classification of $t$-Designs With Group Theoretical Localisation Techniques Applied to Some Steiner Quadruple Systems on 20 Points

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**Abstract:**

The topic of this paper is to determine the isomorphism types of designs which are invariant under a given group. As an example, we consider SQS(20) invariant under a subgroup of the symmetric group $\mathcal{S}_{20}$ isomorphic to the alternating group $\mathcal{A}_5$.

Keywords: t-designs, group action, isomorphism problem, Sylow subgroups.

## 1 Introduction

In order to construct a $t$-design, one can prescribe an automorphism group $A$ for the design; then the incidence matrix between the orbits on $t$-subsets of the point set (as rows) and the orbits on $k$-subsets (as columns) $M_{t,k}^A$ is calculated. The entry in row $i$ and column $j$ indicates the number of $k$-subsets in orbit $j$ containing the representative of the $i$-th orbit on $t$-subsets. We receive designs omitting the prescribed group as an automorphism group as solutions for the diophantine system of equations

$$M_{t,k}^A \cdot x = (\lambda, \dots, \lambda)^T$$

according to the lemma of Kramer and Mesner [10]. The matrix $M_{t,k}^{A}$ is called *Kramer-Mesner matrix*. More information on this method is collected in [1], [15] and [16].

After the construction of designs with certain fixed parameters $t-(v,k,\lambda)$ there arises very naturally the following question:

*Which designs have an identical structure, i.e. which are isomorphic?*

This problem will be the main topic of this paper. A method using some group theory is developed in section 3. As an example, we consider the case of Steiner quadruple systems 3-(20, 4, 1) on 20 points with different prescribed automorphism groups. These quadruple systems are interesting as a step towards the search for 5-(22, 6, 1) designs. Since all known Steiner 4-designs are derived from Steiner 5-designs, it makes sense not to try to construct the Steiner 4-designs 4-(21, 5, 1) in the direct way.

Besides the package DISCRETA [2] for the construction of $t$-designs with prescribed automorphism group we also use the package GAP [17] for the group theoretical computations. More exactly, we wrote an interface between DISCRETA and GAP.

# 2 Notation and basic definitions

The problem of finding isomorphisms between designs and deciding whether two designs are isomorphic or not, can be regarded with the help of group actions. Let $G$ be a finite group acting on a finite set $\Omega$:

$$G \times \Omega \to \Omega, (g,\omega) \mapsto \omega^{g}$$

For group actions the following notations turn out to be relevant:

| | |
|---|---|
| $G\omega := \{\omega^{g} : g \in G\}$ | orbit of $\omega$ under the action of $G$ |
| $Fix_{\Omega}(g) := \{\omega \in \Omega : \omega^{g} = \omega\}$ | set of fixed points of $g$ in $\Omega$ |
| $Fix_{\Omega}(U) := \{\omega \in \Omega : \omega^{g} = \omega \ \forall g \in G\}$ | set of fixed points of $U \leq G$ in $\Omega$ |
| $Stab_{G}(\omega) := \{g \in G : \omega^{g} = \omega\}$ | stabilizer of $\omega$ in $G$ |

Return to the designs: A $t-(v,k,\lambda)$ *design* is a collection of $k$-subsets (called *blocks*) of a $v$-point set $\mathcal{V}$, such that every $t$-subset of $\mathcal{V}$ is contained in exactly $\lambda$ blocks. It is called simple, if every block occurs only once. In this paper, we only consider simple designs. We usually identify the point set of a design without loss of generality with $\mathcal{V} = \{1,\ldots,v\}$. Every design is uniquely caracterized by its block set $\mathcal{B}$ and therefore identified with $\mathcal{B}$.

**2.1 Definition** *Two $t-(v,k,\lambda)$ designs $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic, if there exists $\pi \in \mathcal{S}_v$, such that*

$$\pi(\mathcal{B}) = \mathcal{B}'$$

*where $\pi$ is applied to the elements of each block of $\mathcal{B}$.*

The isomorphism types of the $t-(v,k,\lambda)$ designs arise as the orbits of the following action of $\mathcal{S}_v$ on the set of designs for fixed parameters $t$, $v$, $k$ and $\lambda$ $\mathcal{D}_{t-(v,k,\lambda)} := \{t-(v,k,\lambda) \text{ designs}\}$:

$$\mathcal{S}_v \times \mathcal{D}_{t-(v,k,\lambda)} \to \mathcal{D}_{t-(v,k,\lambda)}, (\pi, \mathcal{B}) \mapsto \mathcal{B}^\pi$$

The full automorphism group of a design is exactly the stabilizer of its block set $\mathcal{B}$ in the symmetric group $\mathcal{S}_v$ on $v$ points. That means: $Aut(\mathcal{B}) = Stab_{\mathcal{S}_v}(\mathcal{B})$. The designs with automorphism group $A$ thus are the fixed points $Fix_{\mathcal{D}_{t-(v,k,\lambda)}}(A)$ of $A$ of this action.

# 3 Group theoretical background

Let us recall basic facts about group actions. In this section let a group $G$ act on a set $\Omega$. In the theory of group actions, we have a basic Lemma, which in combinatorics is mostly called Burnside's Lemma. But according to Neumann [13] (see also [9]), it was already known to Cauchy [5] and Frobenius [6]. Thus, we call this fundamental lemma

**3.1 Lemma of Cauchy-Frobenius** *Let $G$ act on a set $\Omega$. Then*

$$\#\textit{orbits of } G \textit{ on } \Omega \ = \frac{1}{|G|} \cdot \sum_{g \in G} |Fix_\Omega(g)|$$

The lemma yields a connection between orbits and fixed points. But it is not constructive and in case of large groups $G$ not feasible. In case of designs, this would mean that for every permutation $\pi \in \mathcal{S}_v$ all designs fixed by $\pi$ would have to be determined. So better methods to find the orbits should be developed. It turns out, that the immediate equation

$$Stab(\omega^g) = Stab(\omega)^g \tag{1}$$

is very helpful: orbits can be caracterized by stabilizer class. Therefore, for a given representative $\omega$ of an orbit with stabilizer $U := Stab(\omega)$, the orbit of $\omega$ is called an *orbit of type $U$*.

Then one question arises very naturally:

*How many orbits of each type do exist?*

In this paper we want to give some partial answers to this fairly complex problem. The method is developed in two sequel steps:

Step 1: *Burnside's Lemma*

Step 2: *Jordan's Theorem*

### 3.1 Burnside's Lemma

Consider the lattice $\mathcal{L}(G)$ of subgroups of $G$. In particular, $\mathcal{L}(G)$ is a poset with respect to inclusion, denoted as $\leq$. The *zeta-function* of $\mathcal{L}(G)$ is defined as

$$\zeta(U,V) := \begin{cases} 1 \text{ if } U \leq V, \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

for all $U, V \in \mathcal{L}(G)$. The inclusion relations of the elements of $\mathcal{L}(G)$ are collected in the so called *zeta-matrix*

$$\zeta(G) := \left(\zeta(U_i, U_j)\right)_{i,j}.$$

Assume that the subgroups $U_i$ of $G$ are numbered in such a way that

$$U_i \leq U_j \Longrightarrow i \leq j. \tag{3}$$

Then the zeta-matrix is upper triangular with 1's on the diagonal, so it is invertible over the ring of integers. Its inverse is called the *Moebius-matrix* $\mu(G)$ of $G$ [14]. In order to calculate the orbits of type $U$ on the set $\Omega$ for $U \leq G$, consider the set of fixed points $Fix_\Omega(U)$ of $U$ on $\Omega$. The order $|Fix_\Omega(U)|$ was called the *mark* of $U$ on $\Omega$ by Burnside in [4]. Similar to the Lemma of Cauchy-Frobenius, one has

$$|Fix_\Omega(U)| = \sum_{V:U\leq V\leq G} \zeta(U,V)|N_G(V)/V| \cdot \#\text{orbits of type } V.$$

By Moebius-inversion, this equation is equivalent to:

$$\#\text{orbits of type } U = \frac{1}{|N_G(U)/U|} \sum_{V:U\leq V\leq G} \mu(U,V) \cdot |Fix_\Omega(V)|. \tag{4}$$

Expression (4) can be simplified by considering the set of conjugacy classes $\widetilde{\mathcal{L}}(G) := \{\widetilde{U}_1, \ldots, \widetilde{U}_r\}$ of $G$ instead of the lattice $\mathcal{L}(G)$, where each $U_i$ is a representative of conjugacy class $\widetilde{U}_i$. It can be reformulated with the help of the matrix $B(G)$ of $G$ defined by

$$b_{ij} := \frac{1}{|N_G(U_i)/U_i|} \cdot \sum_{V \in \widetilde{U}_j} \mu(U_i, V). \tag{5}$$

Burnside [4] introduced the table of marks and remarked that this matrix is invertible matrix. The inverted matrix now is called *Burnside matrix* $B(G)$. So reformulate equation (4) as

**3.2 Burnside's Lemma** *Let $G$ act on $\Omega$, $\widetilde{\mathcal{L}}(G) := \{\widetilde{U}_1, \ldots, \widetilde{U}_r\}$ the set of conjugacy classes of subgroups of $G$; let $B(G)$ be the Burnside matrix of $G$ w.r.t. the ordering in $\widetilde{\mathcal{L}}$. Then*

$$B(G) \cdot \begin{pmatrix} \vdots \\ |Fix_\Omega(U_j)| \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \#\textit{orbits of type } U_i \\ \vdots \end{pmatrix}.$$

Since $B(G)$ is upper triangular, the evaluation can be restricted to some bottom rows. This lemma can be helpful for the classification of $t$-designs with prescribed automorphism group, when the prescribed groups are quite large. "Large" means that the partial subgroup lattice between the prescribed group $A$ and $\mathcal{S}_v$ is known. In the lattice, only the overgroups of $A$ in $\mathcal{S}_v$ have to be considered and therefore, the relevant Burnside matrix is only a part of the full Burnside matrix $B(\mathcal{S}_v)$. As an example take $PSL_2(27)$ with degree 28, and consider the parameter quadruple $4-(28, 6, 45)$:

**3.3 Example**

*Fig. 1 shows a partial subgroup lattice of $\mathcal{S}_{28}$ with several automorphism groups of 4-(28, 6, 45) designs. The isomorphism problem can be solved with the equation according to Burnside's Lemma 3.2*

$$\begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 13,078,960 \\ 704 \\ 58 \\ 8 \end{pmatrix} = \begin{pmatrix} 2,179,701 \\ 232 \\ 25 \\ 8 \end{pmatrix}$$

*where we left out the groups $\mathcal{A}_v$ and $\mathcal{S}_v$, because no nontrivial design exists with these automorphism groups. Therefore we obtain the informations of Tab. 1.*

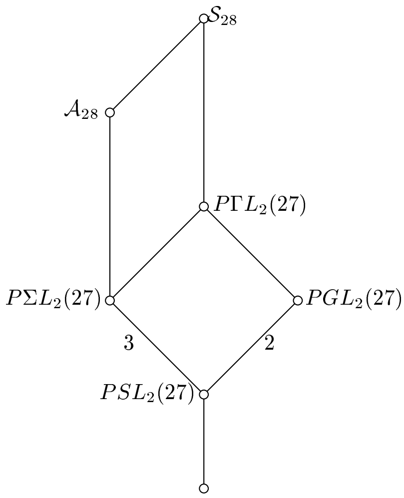



Figure 1: Relevant Partial Subgroup Lattice Above $PSL_2(27)$ and Isomorphism Types of 4-(28, 6, 45). The Numbers Between two Subgroups Denote the Indices.

| Group | Group order | # solutions | # isom. types |
|---|---|---|---|
| $PSL_2(27)$ | 9,828 | 13,078,960 | 2,179,701 |
| $PGL_2(27)$ | 19,656 | 704 | 232 |
| $P\Sigma L_2(27)$ | 29,484 | 58 | 25 |
| $P\Gamma L_2(27)$ | 58,968 | 8 | 8 |
| in total: | | 13,079,730 | 2,179,966 |

Table 1: Isomorphism Types of 4-(28, 6, 45) with Certain Automorphism Groups

In general cases, the relevant partial subgroup lattice is not known. A good example is a question, which was posed by Earl S. Kramer and Dale M. Mesner in 1976 in the seminal paper [10]:

> "No systematic attempt was made to determine the isomorphism types of our designs. For example, in searching for (45; 2, 5, 13)'s [i.e. 2-(13, 5, 45) designs in our notation], using a 2 by 19 matrix, we failed to specify an upper bound on the number of solutions and the computer run was cut short only because it has printed a specified quota of lines but had in the meantime produced 324 solutions (each using column 1) and would likely have found many more. How many of these are nonisomorphic is a question we might be afraid to consider."

Kramer and Mesner had used a subgroup of $Hol(C_{13})$ of order 78, which is isomorphic to $U_3 := C_{13} \ltimes C_6$. We show that we need not know the whole partial subgroup lattice of $\mathcal{S}_{13}$ with respect to $U_3$ to solve the isomorphism problem with the help of a lemma due to C. Jordan [8], see also [7] and [18]:

**3.4 Jordan's Lemma** *Let $P$ be a p-subgroup of $G$ for fixed prime p, $A \leq G$ and $\Delta := Fix_\Omega(A)$. Let $\Delta'$ be a subset of $\Delta$ such that $P \in Syl_p\,(Stab(\delta))$ for all $\delta \in \Delta'$.*

*If for $\delta_1, \delta_2 \in \Delta'$ there exists some $g \in G$ with $\delta_1^g = \delta_2$, then*

$$\exists \quad n \in N_G(P): \qquad \delta_1^n = \delta_2$$

**3.5 Example**
*This method applies to all subgroups $A$, such that $P = C_{13} \leq A \leq Hol(C_{13})$, in particular the case considered by Kramer and Mesner. We have a group $U_1$ isomorphic to $C_{13} \ltimes C_2$ with 13-Sylow subgroup $P = C_{13}$. According to 3.4, we can reduce the search for isomorphisms between the 136.976.801 designs with automorphism group $U_1$ to $N_{\mathcal{S}_{13}}(P) = Hol(C_{13})$.*

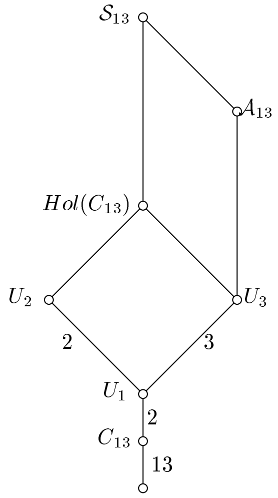



Figure 2: Partial Subgroup Lattice of $\mathcal{S}_{13}$ Relevant for 2-(13, 5, 72). The Numbers Between two Subgroups Denote the Indices.

*Formally the computation for the partial subgroup lattice Fig. 2 of $\mathcal{S}_{13}$ between $A$ and $Hol(C_{13})$ is as in Burnsides Lemma 3.2, but not all overgroups need to be considered.*

$$\begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 136,876,801 \\ 24,643 \\ 890 \\ 28 \end{pmatrix} = \begin{pmatrix} 22,825,216 \\ 8,205 \\ 431 \\ 28 \end{pmatrix}$$

*Therefore we receive the informations collected in Tab. 2.*

| Group | Group order | # solutions | # isom. types |
|---|---|---|---|
| $U_1$ | 26 | 136,876,801 | 22,825,216 |
| $U_2$ | 52 | 24,643 | 8,205 |
| $U_3$ | 78 | 890 | 431 |
| $Hol(C_{13})$ | 156 | 28 | 28 |
| in total: | | 136,902,362 | 22,833,880 |

Table 2: Isomorphism Types of Certain 2-(13, 5, 72)

*Recall that the group considered by Kramer and Mesner is group $U_3$. In each case we need not know whether the prescribed group is the full automorphism group of the designs counted.*

Also in cases, where it can be shown that no design exists for any $p$-group $Q, P < Q$, this lemma is applicable. Nevertheless, the approach is problematic, when $A \not\leq N_G(P)$: then $Fix_\Omega(A)$ in general is not closed under the action of $N_G(P)$ and it is not appropriate to just form the orbits of $N_G(P)$. But the following remark gives us a hint to the solution:

**3.6 Remark** *Let $A \leq G$, $P \in Syl_p(A)$ and $\delta_1, \delta_2 \in Fix_\Omega(A) =: \Delta$. If there exists some $g \in N_G(P), g \notin N_{N_G(P)}(A)$ such that $\delta_1^g = \delta_2$, then*

$$A < \langle A, A^g \rangle \leq Stab(\delta_2).$$

**Proof:** *$A \leq Stab(\delta_1)$ and since $g \notin N_G(A)$, we have $A \neq A^g \leq (Stab(\delta_1))^g = Stab(\delta_2)$. As $A$ is also subgroup of $Stab(\delta_2)$, this yields the claim.*

This is a slightly more general version of [3]. The bigger automorphism groups that have to be tested now can be directly constructed and need not to be known from some catalogue. So the following algorithm can be applied:

**3.7 Algorithm** *Given a group $A \leq G$ and $\Delta := Fix_\Omega(A)$.*

*i) Fix a prime p dividing $|A|$ and compute a $P \in Syl_p(A)$.*

*ii) Compute $N_G(P), N_A(P)$ and $I := N_{N_G(P)}(A)$.*

*iii) Consider a transversal $T$ of $I$ in $N_G(P)$:*
*For all $B := \langle A, A^g \rangle$, where $g \in T$, remove $Fix_\Omega(B)$ from $\Delta$.*

*iv) If for the remaining part of $\Delta$ $P$ is known to be a Sylow subgroup of the stabilizers of the elements in $\Delta$, then determine the orbits of $I$ on this set.*

*v) Determine and output the orbits of $N_G(A)$ on this set.*

*vi) Apply i) - v) on the groups constructed in iii) and their fixed points.*

**3.8 Remark** *Recall that step iii) can be improved by noticing, that $\mathcal{A}_v$ and $\mathcal{S}_v$ can arise only as automorphism groups of trivial designs.*

# 4 Isomorphism classification of 3-(20, 4, 1) designs

There exists exactly one $5-(22,\{6,7,8\},1)$ design with $M_{22}$ as automorphism group. This tBD is "derived" from the famous $5-(24,8,1)$ Witt-design with automorphism group $M_{24}$ in the following way: first skip the point 24 and consider its stabilizer (which is $M_{23}$) and then once again skip point 23 and take its stabilizer $M_{22}$ as automorphism group with generators

$$(1,22,8,19,14)(2,16,5,13,3)(4,11,20,21,17)(6,18,7,12,15)$$
$$(1,14)(2,5,17,3)(4,15,7,9,18,6,20,19)(8,21,16,13,22,10,11,12)$$

The blocks of the Witt design form one orbit under $M_{24}$. During this procedure, the blocks containing the points 23 and/or 24 are shortened from 8-subsets to 6- resp. 7-subsets of the point set.

This design can be constructed with the Kramer-Mesner method: collect the three KM-matrices of $A := M_{22}$ between $5-/6-$subsets resp. $5-/7-$subsets and $5-/8-$subsets into one big Kramer-Mesner matrix $M^A = (M^A_{5,6}|M^A_{5,7}|M^A_{5,8})$ shown in Tab. 3.

The arrows indicate, that every element of the first orbit on 5-subsets (canonical representative: $\{1,2,3,4,5\}$) is contained in exactly one element of the last orbit $\mathcal{O}_1$ on 8-subsets, every element of the second orbit on 5-subsets (canonical representative: $\{1,2,3,4,6\}$) in exactly one element of the seventh orbit $\mathcal{O}_2$ on 7-subsets, every element of the third orbit on 5-subsets (canonical representative: $\{1,2,3,4,7\}$) in exactly one element of the last orbit $\mathcal{O}_3$ on 7-subsets and finally every element of the last orbit on 5-subsets (canonical representative: $\{1,2,3,5,14\}$) in exactly one element of the last orbit $\mathcal{O}_4$ on 6-subsets. So, the

| | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 3 | 2 | 0 | 0 | 0 | 60 | 48 | 12 | 12 | 3 | 1 | 0 | 0 | 156 | 60 | 108 | 156 | 86 | 84 | 21 | 4 | 4 | 1 |
| 15 | 0 | 0 | 2 | 0 | 0 | 60 | 30 | 30 | 15 | 0 | 0 | 1 | 0 | 140 | 60 | 150 | 180 | 60 | 60 | 10 | 15 | 5 | 0 |
| 15 | 0 | 0 | 0 | 2 | 0 | 60 | 30 | 15 | 30 | 0 | 0 | 0 | 1 | 180 | 60 | 150 | 140 | 60 | 60 | 10 | 5 | 15 | 0 |
| 0 | 0 | 16 | 0 | 0 | 1 | 0 | 120 | 0 | 0 | 0 | 16 | 0 | 0 | 160 | 0 | 0 | 160 | 80 | 160 | 120 | 0 | 0 | 0 |
| | | | | | $\uparrow$ | | | | | | | $\uparrow$ | $\uparrow$ | | | | | | | | | | $\uparrow$ |

Table 3: Combined Kramer-Mesner matrix $M^{M_{22}} = (M_{5,6}^{M_{22}} | M_{5,7}^{M_{22}} | M_{5,8}^{M_{22}})$

| Orbit | $k$ | canonical representative | orbit length | stabilizer order |
|---|---|---|---|---|
| $\mathcal{O}_1$ | 8 | $\{1,2,3,4,5,8,11,13\}$ | 330 | 1,344 |
| $\mathcal{O}_2$ | 7 | $\{1,2,3,4,6,15,18\}$ | 176 | 2,520 |
| $\mathcal{O}_3$ | 7 | $\{1,2,3,4,7,10,12\}$ | 176 | 2,520 |
| $\mathcal{O}_4$ | 6 | $\{1,2,3,5,14,17\}$ | 77 | 5,760 |

Table 4: Orbits of $M_{22}$ on 6-, 7- and 8-Subsets

collection of these four orbits in fact gives a 5-design with $\lambda = 1$. Detailed information gives Tab. 4. This recipe can generally be used to obtain tBD's from $t$-designs.

The main challenge is to find $5-(22,k,1)$ designs with only one value $k$, for example $k = 6$. A 3-(20, 4, 1) design still is the best known approximation of a $5-(22,k,1)$ design. $3-(20,4,1)$ designs with various automorphism groups can be considered, even with fairly small automorphism groups as for example the symmetry group of the dodecahedron of order 60, which is isomorphic to $\mathcal{A}_5$. But too small groups lead to isomorphism problems that still are inaccessible by our methods. If the holomorph of the cyclic group $\mathcal{C}_6$, induced on the 3-subsets of 6 points is prescribed (this group has order 12), we receive 704,976 designs. Furthermore we know from [12], that there exist altogether at least $10^{17}$ isomorphism types of SQS on 20 points.

## 4.1 The symmetry group of the dodecahedron

When looking at the dodecahedron of Fig. 3, notice that its symmetry group $A$

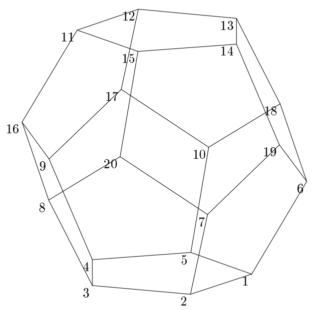



Figure 3: Labeled Dodecahedron

can be generated by the 5-cycles

| | | | |
|---|---|---|---|
| $O_1 : \{1,2,3,4\}_{60}$ | $O_{26} : \{1,2,4,17\}_{60}$ | $O_{51} : \{1,2,10,13\}_{60}$ | $O_{76} : \{1,3,9,20\}_{60}$ |
| $O_2 : \{1,2,3,6\}_{60}$ | $O_{27} : \{1,2,4,18\}_{20}$ | $O_{52} : \{1,2,10,15\}_{30}$ | $O_{77} : \{1,3,10,11\}_{60}$ |
| $O_3 : \{1,2,3,7\}_{60}$ | $O_{28} : \{1,2,4,19\}_{60}$ | $O_{53} : \{1,2,10,16\}_{60}$ | $O_{78} : \{1,3,10,12\}_{30}$ |
| $O_4 : \{1,2,3,8\}_{60}$ | $O_{29} : \{1,2,4,20\}_{60}$ | $O_{54} : \{1,2,10,20\}_{60}$ | $O_{79} : \{1,3,10,14\}_{30}$ |
| $O_5 : \{1,2,3,9\}_{60}$ | $O_{30} : \{1,2,8,9\}_{60}$ | $O_{55} : \{1,2,11,12\}_{30}$ | $O_{80} : \{1,3,10,15\}_{60}$ |
| $O_6 : \{1,2,3,10\}_{60}$ | $O_{31} : \{1,2,8,10\}_{20}$ | $O_{56} : \{1,2,11,13\}_{60}$ | $O_{81} : \{1,3,10,16\}_{60}$ |
| $O_7 : \{1,2,3,11\}_{60}$ | $O_{32} : \{1,2,8,11\}_{60}$ | $O_{57} : \{1,2,11,15\}_{60}$ | $O_{82} : \{1,3,10,19\}_{30}$ |
| $O_8 : \{1,2,3,12\}_{60}$ | $O_{33} : \{1,2,8,12\}_{20}$ | $O_{58} : \{1,2,11,16\}_{60}$ | $O_{83} : \{1,3,11,13\}_{60}$ |
| $O_9 : \{1,2,3,13\}_{60}$ | $O_{34} : \{1,2,8,13\}_{60}$ | $O_{59} : \{1,2,11,17\}_{60}$ | $O_{84} : \{1,3,11,14\}_{60}$ |
| $O_{10} : \{1,2,3,14\}_{60}$ | $O_{35} : \{1,2,8,14\}_{60}$ | $O_{60} : \{1,2,13,15\}_{30}$ | $O_{85} : \{1,3,11,17\}_{30}$ |
| $O_{11} : \{1,2,3,15\}_{60}$ | $O_{36} : \{1,2,8,15\}_{60}$ | $O_{61} : \{1,2,13,16\}_{30}$ | $O_{86} : \{1,3,11,18\}_{30}$ |
| $O_{12} : \{1,2,3,16\}_{60}$ | $O_{37} : \{1,2,8,16\}_{60}$ | $O_{62} : \{1,2,13,17\}_{60}$ | $O_{87} : \{1,3,12,14\}_{60}$ |
| $O_{13} : \{1,2,3,17\}_{60}$ | $O_{38} : \{1,2,8,17\}_{60}$ | $O_{63} : \{1,2,15,17\}_{60}$ | $O_{88} : \{1,3,12,15\}_{30}$ |
| $O_{14} : \{1,2,3,18\}_{60}$ | $O_{39} : \{1,2,8,18\}_{20}$ | $O_{64} : \{1,3,7,9\}_{30}$ | $O_{89} : \{1,3,12,16\}_{30}$ |
| $O_{15} : \{1,2,3,19\}_{30}$ | $O_{40} : \{1,2,9,10\}_{60}$ | $O_{65} : \{1,3,7,10\}_{60}$ | $O_{90} : \{1,3,13,15\}_{60}$ |
| $O_{16} : \{1,2,3,20\}_{60}$ | $O_{41} : \{1,2,9,11\}_{60}$ | $O_{66} : \{1,3,7,11\}_{60}$ | $O_{91} : \{1,3,13,17\}_{30}$ |
| $O_{17} : \{1,2,4,8\}_{60}$ | $O_{42} : \{1,2,9,12\}_{60}$ | $O_{67} : \{1,3,7,12\}_{60}$ | $O_{92} : \{1,3,14,16\}_{5}$ |
| $O_{18} : \{1,2,4,9\}_{30}$ | $O_{43} : \{1,2,9,13\}_{60}$ | $O_{68} : \{1,3,9,10\}_{30}$ | $O_{93} : \{1,3,14,17\}_{30}$ |
| $O_{19} : \{1,2,4,10\}_{60}$ | $O_{44} : \{1,2,9,14\}_{60}$ | $O_{69} : \{1,3,9,11\}_{60}$ | $O_{94} : \{1,3,15,17\}_{30}$ |
| $O_{20} : \{1,2,4,11\}_{60}$ | $O_{45} : \{1,2,9,15\}_{60}$ | $O_{70} : \{1,3,9,12\}_{60}$ | $O_{95} : \{1,8,14,17\}_{5}$ |
| $O_{21} : \{1,2,4,12\}_{60}$ | $O_{46} : \{1,2,9,16\}_{60}$ | $O_{71} : \{1,3,9,13\}_{60}$ | $O_{96} : \{1,9,13,20\}_{12}$ |
| $O_{22} : \{1,2,4,13\}_{60}$ | $O_{47} : \{1,2,9,17\}_{60}$ | $O_{72} : \{1,3,9,14\}_{60}$ | |
| $O_{23} : \{1,2,4,14\}_{60}$ | $O_{48} : \{1,2,9,20\}_{30}$ | $O_{73} : \{1,3,9,15\}_{60}$ | |
| $O_{24} : \{1,2,4,15\}_{60}$ | $O_{49} : \{1,2,10,11\}_{60}$ | $O_{74} : \{1,3,9,18\}_{30}$ | |
| $O_{25} : \{1,2,4,16\}_{60}$ | $O_{50} : \{1,2,10,12\}_{60}$ | $O_{75} : \{1,3,9,19\}_{60}$ | |

Table 5: Orbits of the Symmetry Group of the Dodecahedron on 4-Subsets of the Vertices

$$(1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20),$$
$$(1,2,7,19,6)(3,20,14,18,5)(4,8,15,13,10)(9,16,11,12,17).$$

Thus, a group of order 60 and degree 20 is obtained, which is isomorphic to the alternating group $\mathcal{A}_5$, embedded into $G := \mathcal{S}_{20}$. This group acts on the 4-subsets of the vertices of the dodecahedron. They fall into the 96 orbits shown in Tab. 5 with their canonical representatives (the index shows the orbit length).

The Kramer-Mesner-matrix $M^A_{3,4}$ is of size 21 × 96. The 0/1-vectors solving the diophantine system of equations

$$M^A_{3,4} \cdot x = (1,\dots,1)^T$$

represent $3-(20,4,1)$ designs. We obtain the 152 solutions of Tab. 6 and Tab. 7. In this notation, the numbers represent the chosen 4-orbits of Tab. 5.

$\mathcal{B}_{1}$ : $\{3, 23, 37, 55, 70, 79, 95, 96\}$
$\mathcal{B}_{2}$ : $\{3, 23, 37, 55, 73, 78, 95, 96\}$
$\mathcal{B}_{3}$ : $\{3, 23, 37, 55, 72, 81, 95, 96\}$
$\mathcal{B}_{4}$ : $\{3, 23, 37, 55, 74, 80, 95, 96\}$
$\mathcal{B}_{5}$ : $\{15, 40, 55, 60, 67, 86, 95, 96\}$
$\mathcal{B}_{6}$ : $\{16, 30, 55, 60, 67, 86, 95, 96\}$
$\mathcal{B}_{7}$ : $\{6, 27, 55, 60, 67, 86, 95, 96\}$
$\mathcal{B}_{8}$ : $\{5, 29, 55, 60, 67, 86, 95, 96\}$
$\mathcal{B}_{9}$ : $\{12, 23, 36, 55, 67, 73, 95, 96\}$
$\mathcal{B}_{10}$ : $\{3, 22, 36, 40, 55, 86, 95, 96\}$
$\mathcal{B}_{11}$ : $\{14, 23, 51, 55, 67, 79, 95, 96\}$
$\mathcal{B}_{12}$ : $\{3, 24, 30, 51, 55, 86, 95, 96\}$
$\mathcal{B}_{13}$ : $\{3, 18, 55, 62, 69, 92, 95, 96\}$
$\mathcal{B}_{14}$ : $\{3, 18, 55, 62, 75, 92, 95, 96\}$
$\mathcal{B}_{15}$ : $\{3, 18, 55, 62, 68, 83, 92, 95, 96\}$
$\mathcal{B}_{16}$ : $\{13, 18, 55, 67, 69, 92, 95, 96\}$
$\mathcal{B}_{17}$ : $\{13, 18, 55, 67, 75, 92, 95, 96\}$
$\mathcal{B}_{18}$ : $\{13, 18, 55, 67, 68, 83, 92, 95, 96\}$
$\mathcal{B}_{19}$ : $\{3, 26, 34, 40, 55, 84, 90, 95, 96\}$
$\mathcal{B}_{20}$ : $\{10, 26, 39, 54, 55, 67, 73, 95, 96\}$
$\mathcal{B}_{21}$ : $\{3, 25, 30, 52, 55, 84, 90, 95, 96\}$
$\mathcal{B}_{22}$ : $\{2, 25, 30, 55, 63, 67, 84, 90, 95, 96\}$
$\mathcal{B}_{23}$ : $\{3, 25, 30, 54, 55, 63, 84, 88, 95, 96\}$
$\mathcal{B}_{24}$ : $\{11, 25, 39, 54, 55, 67, 79, 95, 96\}$
$\mathcal{B}_{25}$ : $\{3, 25, 39, 47, 54, 55, 79, 88, 95, 96\}$
$\mathcal{B}_{26}$ : $\{2, 25, 39, 47, 55, 67, 79, 90, 95, 96\}$
$\mathcal{B}_{27}$ : $\{4, 26, 46, 54, 55, 67, 73, 84, 95, 96\}$
$\mathcal{B}_{28}$ : $\{4, 26, 40, 55, 61, 67, 84, 90, 95, 96\}$
$\mathcal{B}_{29}$ : $\{4, 26, 44, 51, 55, 67, 84, 91, 95, 96\}$
$\mathcal{B}_{30}$ : $\{3, 26, 39, 40, 55, 61, 87, 90, 95, 96\}$
$\mathcal{B}_{31}$ : $\{3, 26, 39, 46, 54, 55, 73, 87, 95, 96\}$
$\mathcal{B}_{32}$ : $\{3, 26, 39, 44, 51, 55, 87, 91, 95, 96\}$
$\mathcal{B}_{33}$ : $\{2, 28, 36, 46, 55, 67, 73, 83, 95, 96\}$
$\mathcal{B}_{34}$ : $\{3, 28, 36, 40, 55, 61, 83, 88, 95, 96\}$
$\mathcal{B}_{35}$ : $\{3, 28, 37, 44, 55, 68, 84, 88, 91, 95, 96\}$
$\mathcal{B}_{36}$ : $\{2, 28, 39, 44, 55, 60, 67, 68, 91, 95, 96\}$
$\mathcal{B}_{37}$ : $\{3, 28, 31, 44, 55, 62, 69, 95, 96\}$
$\mathcal{B}_{38}$ : $\{3, 28, 31, 44, 55, 62, 75, 95, 96\}$
$\mathcal{B}_{39}$ : $\{3, 28, 31, 44, 55, 62, 68, 83, 95, 96\}$
$\mathcal{B}_{40}$ : $\{13, 28, 31, 44, 55, 67, 69, 95, 96\}$
$\mathcal{B}_{41}$ : $\{13, 28, 31, 44, 55, 67, 75, 95, 96\}$
$\mathcal{B}_{42}$ : $\{13, 28, 31, 44, 55, 67, 68, 83, 95, 96\}$
$\mathcal{B}_{43}$ : $\{4, 28, 47, 51, 55, 67, 79, 83, 95, 96\}$
$\mathcal{B}_{44}$ : $\{3, 28, 30, 51, 55, 63, 83, 87, 95, 96\}$
$\mathcal{B}_{45}$ : $\{2, 25, 36, 44, 55, 67, 85, 90, 95, 96\}$
$\mathcal{B}_{46}$ : $\{3, 25, 36, 44, 54, 55, 85, 88, 95, 96\}$
$\mathcal{B}_{47}$ : $\{4, 28, 44, 54, 55, 60, 67, 68, 85, 95, 96\}$
$\mathcal{B}_{48}$ : $\{3, 28, 37, 44, 55, 68, 85, 87, 90, 95, 96\}$
$\mathcal{B}_{49}$ : $\{12, 21, 30, 63, 67, 76, 90, 95\}$
$\mathcal{B}_{50}$ : $\{12, 20, 30, 63, 67, 76, 84, 95\}$
$\mathcal{B}_{51}$ : $\{10, 29, 32, 63, 67, 76, 90, 95\}$
$\mathcal{B}_{52}$ : $\{3, 22, 30, 50, 63, 76, 84, 95\}$
$\mathcal{B}_{53}$ : $\{9, 18, 63, 67, 76, 81, 84, 95\}$
$\mathcal{B}_{54}$ : $\{3, 18, 56, 63, 76, 81, 84, 95\}$
$\mathcal{B}_{55}$ : $\{12, 20, 39, 47, 67, 76, 79, 95\}$
$\mathcal{B}_{56}$ : $\{3, 22, 32, 47, 54, 76, 79, 95\}$
$\mathcal{B}_{57}$ : $\{3, 22, 39, 47, 50, 76, 79, 95\}$
$\mathcal{B}_{58}$ : $\{3, 21, 34, 47, 54, 76, 81, 95\}$
$\mathcal{B}_{59}$ : $\{3, 29, 39, 47, 56, 76, 92, 95\}$
$\mathcal{B}_{60}$ : $\{9, 29, 39, 47, 67, 76, 92, 95\}$
$\mathcal{B}_{61}$ : $\{4, 21, 46, 54, 63, 67, 76, 81, 95\}$
$\mathcal{B}_{62}$ : $\{4, 29, 46, 57, 67, 76, 83, 92, 95\}$
$\mathcal{B}_{63}$ : $\{4, 21, 47, 54, 61, 67, 76, 81, 95\}$
$\mathcal{B}_{64}$ : $\{4, 29, 47, 58, 67, 76, 83, 92, 95\}$
$\mathcal{B}_{65}$ : $\{14, 29, 44, 58, 67, 76, 84, 90, 95\}$
$\mathcal{B}_{66}$ : $\{3, 24, 39, 44, 54, 58, 76, 81, 95\}$
$\mathcal{B}_{67}$ : $\{3, 29, 32, 47, 61, 76, 87, 90, 95\}$
$\mathcal{B}_{68}$ : $\{3, 18, 57, 61, 76, 81, 83, 87, 95\}$
$\mathcal{B}_{69}$ : $\{3, 29, 32, 46, 63, 76, 87, 90, 95\}$
$\mathcal{B}_{70}$ : $\{3, 18, 58, 63, 76, 81, 83, 87, 95\}$
$\mathcal{B}_{71}$ : $\{3, 28, 36, 46, 49, 76, 88, 91, 95\}$
$\mathcal{B}_{72}$ : $\{2, 28, 39, 46, 57, 67, 76, 79, 90, 95\}$
$\mathcal{B}_{73}$ : $\{3, 28, 39, 46, 54, 57, 76, 79, 88, 95\}$
$\mathcal{B}_{74}$ : $\{2, 28, 36, 42, 61, 67, 76, 91, 95\}$
$\mathcal{B}_{75}$ : $\{3, 28, 30, 54, 57, 61, 76, 84, 88, 95\}$
$\mathcal{B}_{76}$ : $\{2, 28, 30, 57, 61, 67, 76, 84, 90, 95\}$

Table 6: Designs Invariant Under the Symmetry Group of the Dodecahedron (Part I)

We discuss how our methods can be used to solve the isomorphism problem for these designs considering a 5-Sylow subgroup $P$ of $A$. It can be verified that no overgroup of $P$ of order 25 is admitted as an automorphism group of a 3-(20, 4, 1). So apply the algorithm: Compute the transversal $T$ of $H := N_{N_G(A)}(P)$ in $N_G(P)$ of length 1,500. From $T$, the groups $\langle A, A^g \rangle$ with $g \in T$ can be constructed. There are 13 groups with $P$ as Sylow subgroup (which are **not** $\mathcal{A}_{20}$ or $\mathcal{S}_{20} = G$) falling into 7 conjugacy classes under the action of $N_G(P)$. Actually, two of the overgroups (and of course their conjugates) turn out to be automorphism groups of some designs, as the partial subgroup lattice 4 shows.

$\mathcal{B}_{77} : \{3, 28, 30, 52, 58, 76, 84, 90, 95\}$
$\mathcal{B}_{78} : \{2, 28, 30, 58, 63, 67, 76, 84, 90, 95\}$
$\mathcal{B}_{79} : \{3, 28, 30, 54, 58, 63, 76, 84, 88, 95\}$
$\mathcal{B}_{80} : \{11, 28, 39, 54, 58, 67, 76, 79, 95\}$
$\mathcal{B}_{81} : \{3, 28, 39, 47, 54, 58, 76, 79, 88, 95\}$
$\mathcal{B}_{82} : \{2, 28, 39, 47, 58, 67, 76, 79, 90, 95\}$
$\mathcal{B}_{83} : \{15, 46, 54, 57, 67, 76, 85, 92, 95\}$
$\mathcal{B}_{84} : \{3, 18, 57, 61, 76, 78, 85, 90, 95\}$
$\mathcal{B}_{85} : \{3, 18, 58, 63, 76, 78, 85, 90, 95\}$
$\mathcal{B}_{86} : \{15, 47, 54, 58, 67, 76, 85, 92, 95\}$
$\mathcal{B}_{87} : \{3, 20, 37, 44, 76, 78, 85, 95\}$
$\mathcal{B}_{88} : \{15, 44, 50, 60, 67, 76, 85, 95\}$
$\mathcal{B}_{89} : \{12, 20, 36, 44, 67, 76, 85, 95\}$
$\mathcal{B}_{90} : \{3, 22, 36, 44, 50, 76, 85, 95\}$
$\mathcal{B}_{91} : \{3, 26, 31, 44, 56, 76, 85, 95\}$
$\mathcal{B}_{92} : \{9, 26, 31, 44, 67, 76, 85, 95\}$
$\mathcal{B}_{93} : \{3, 28, 37, 41, 76, 80, 85, 95\}$
$\mathcal{B}_{94} : \{6, 28, 33, 60, 67, 76, 85, 95\}$
$\mathcal{B}_{95} : \{2, 28, 36, 44, 58, 67, 76, 85, 90, 95\}$
$\mathcal{B}_{96} : \{3, 28, 36, 44, 54, 58, 76, 85, 88, 95\}$
$\mathcal{B}_{97} : \{3, 18, 55, 62, 76, 82, 83, 92\}$
$\mathcal{B}_{98} : \{13, 18, 55, 67, 76, 82, 83, 92\}$
$\mathcal{B}_{99} : \{3, 28, 37, 44, 55, 76, 82, 84, 88, 91\}$
$\mathcal{B}_{100} : \{2, 28, 39, 44, 55, 60, 67, 76, 82, 91\}$
$\mathcal{B}_{101} : \{3, 28, 31, 44, 55, 62, 76, 82, 83\}$
$\mathcal{B}_{102} : \{13, 28, 31, 44, 55, 67, 76, 82, 83\}$
$\mathcal{B}_{103} : \{4, 28, 44, 54, 55, 60, 67, 76, 82, 85\}$
$\mathcal{B}_{104} : \{3, 28, 37, 44, 55, 76, 82, 85, 87, 90\}$
$\mathcal{B}_{105} : \{3, 27, 46, 54, 59, 82, 92, 96\}$
$\mathcal{B}_{106} : \{7, 27, 46, 54, 67, 82, 92, 96\}$
$\mathcal{B}_{107} : \{14, 21, 46, 54, 67, 73, 82, 96\}$
$\mathcal{B}_{108} : \{3, 24, 39, 46, 50, 73, 82, 96\}$
$\mathcal{B}_{109} : \{3, 24, 32, 46, 54, 73, 82, 96\}$
$\mathcal{B}_{110} : \{3, 20, 39, 46, 52, 74, 82, 96\}$
$\mathcal{B}_{111} : \{2, 20, 39, 46, 63, 67, 74, 82, 96\}$
$\mathcal{B}_{112} : \{2, 27, 46, 57, 67, 82, 83, 92, 96\}$
$\mathcal{B}_{113} : \{16, 39, 46, 57, 67, 82, 91, 92, 96\}$
$\mathcal{B}_{114} : \{3, 27, 46, 50, 63, 82, 84, 88, 96\}$
$\mathcal{B}_{115} : \{7, 18, 61, 67, 74, 82, 90, 96\}$
$\mathcal{B}_{116} : \{3, 18, 59, 61, 74, 82, 90, 96\}$
$\mathcal{B}_{117} : \{14, 20, 40, 61, 67, 82, 84, 96\}$
$\mathcal{B}_{118} : \{14, 21, 40, 61, 67, 82, 90, 96\}$
$\mathcal{B}_{119} : \{11, 27, 50, 61, 67, 82, 84, 96\}$
$\mathcal{B}_{120} : \{3, 24, 32, 40, 61, 82, 90, 96\}$
$\mathcal{B}_{121} : \{3, 27, 47, 50, 61, 82, 84, 88, 96\}$
$\mathcal{B}_{122} : \{3, 18, 57, 61, 74, 82, 83, 88, 96\}$
$\mathcal{B}_{123} : \{3, 18, 57, 61, 70, 82, 84, 91, 96\}$
$\mathcal{B}_{124} : \{2, 20, 39, 47, 61, 67, 74, 82, 96\}$
$\mathcal{B}_{125} : \{3, 18, 58, 63, 74, 82, 83, 88, 96\}$
$\mathcal{B}_{126} : \{3, 18, 58, 63, 70, 82, 84, 91, 96\}$
$\mathcal{B}_{127} : \{2, 27, 47, 58, 67, 82, 83, 92, 96\}$
$\mathcal{B}_{128} : \{16, 39, 47, 58, 67, 82, 91, 92, 96\}$
$\mathcal{B}_{129} : \{3, 21, 37, 44, 70, 82, 91, 96\}$
$\mathcal{B}_{130} : \{16, 32, 44, 60, 67, 82, 91, 96\}$
$\mathcal{B}_{131} : \{14, 21, 44, 51, 67, 82, 91, 96\}$
$\mathcal{B}_{132} : \{3, 24, 32, 44, 51, 82, 91, 96\}$
$\mathcal{B}_{133} : \{7, 25, 31, 44, 67, 82, 91, 96\}$
$\mathcal{B}_{134} : \{3, 25, 31, 44, 59, 82, 91, 96\}$
$\mathcal{B}_{135} : \{12, 27, 44, 57, 67, 82, 84, 90, 96\}$
$\mathcal{B}_{136} : \{3, 22, 39, 44, 54, 57, 74, 82, 96\}$
$\mathcal{B}_{137} : \{3, 28, 34, 40, 57, 82, 84, 90, 96\}$
$\mathcal{B}_{138} : \{10, 28, 39, 54, 57, 67, 73, 82, 96\}$
$\mathcal{B}_{139} : \{3, 28, 37, 42, 72, 82, 91, 96\}$
$\mathcal{B}_{140} : \{5, 28, 49, 60, 67, 82, 91, 96\}$
$\mathcal{B}_{141} : \{4, 28, 46, 54, 57, 67, 73, 82, 84, 96\}$
$\mathcal{B}_{142} : \{4, 28, 40, 57, 61, 67, 82, 84, 90, 96\}$
$\mathcal{B}_{143} : \{4, 28, 47, 54, 58, 67, 73, 82, 84, 96\}$
$\mathcal{B}_{144} : \{4, 28, 40, 58, 63, 67, 82, 84, 90, 96\}$
$\mathcal{B}_{145} : \{4, 28, 44, 51, 57, 67, 82, 84, 91, 96\}$
$\mathcal{B}_{146} : \{3, 28, 39, 40, 57, 61, 82, 87, 90, 96\}$
$\mathcal{B}_{147} : \{3, 28, 39, 46, 54, 57, 73, 82, 87, 96\}$
$\mathcal{B}_{148} : \{3, 28, 39, 40, 58, 63, 82, 87, 90, 96\}$
$\mathcal{B}_{149} : \{3, 28, 39, 47, 54, 58, 73, 82, 87, 96\}$
$\mathcal{B}_{150} : \{3, 28, 39, 44, 51, 57, 82, 87, 91, 96\}$
$\mathcal{B}_{151} : \{4, 28, 41, 51, 63, 67, 82, 85, 96\}$
$\mathcal{B}_{152} : \{3, 28, 33, 47, 51, 82, 85, 87, 96\}$

Table 7: Designs Invariant Under the Symmetry Group of the Dodecahedron (Part II)

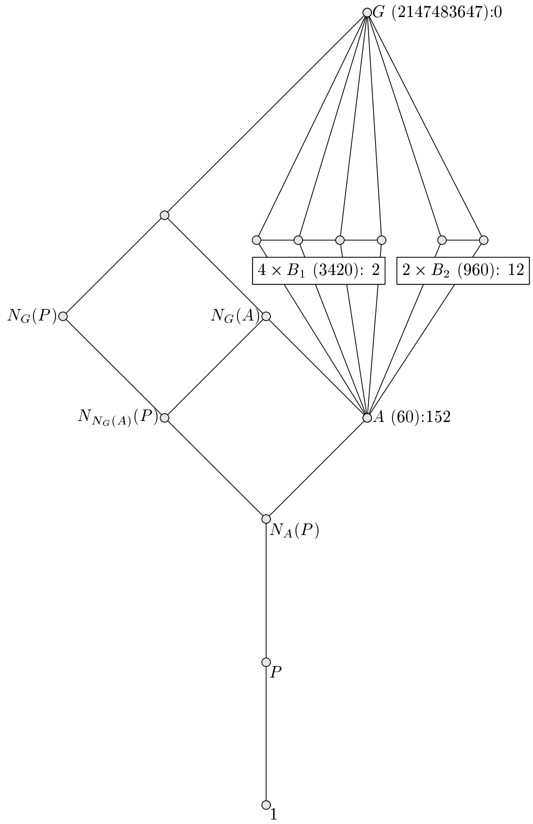



Figure 4: Partial Subgroup Lattice of $\mathcal{S}_{20}$ Relevant for Certain SQS(20)

The notation $4 \times B_1(3420) : 2$ indicates that the conjugacy class of group $B_1$ of order 3,420 has length 4, and each of them is automorphism group of 2 designs. So let us have a look at the overgroups.

### 4.1.1 Group $B_1$

Group $B_1$ is generated by

$$
\begin{array}{l}
(1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20) \\
(1,2,7,19,6)(3,20,14,18,5)(4,8,15,13,10)(9,16,11,12,17) \\
(1,2,13,19,17)(3,4,11,14,20)(5,7,9,8,10)(6,18,12,16,15) \\
(1,2,15,4,16)(3,17,13,11,8)(5,7,20,18,14)(6,10,9,12,19)
\end{array}
$$

and is perfect of order 3.420. The orbits of $A$ on 4-sets are fused to orbits of $B_1$. We describe the fusion by a mapping:

$$
\begin{array}{r}
\{1,2,6,9,15,18,24,25,30,32,33,38,41,42,45,51,52,53,57,61, \\
62,65,66,69,73,74,75,84,88,89,94\} \mapsto 1 \\
\{3,21,34,47,54,76,81,95\} \mapsto 2 \\
\{4,7,12,19,23,26,31,35,43,55,56,63,68,70,79,85,87,90\} \mapsto 3 \\
\{5,28,49,60,67,82,91,96\} \mapsto 4 \\
\{8,10,11,13,14,16,17,20,22,27,29,36,37,39,40,44,46,48,50,58, \\
59,64,71,72,77,78,80,83,86,92,93\} \mapsto 5
\end{array}
$$

A solution admits an overgroup as automorphism group, if the solution vector calculated above under the group $A$ is constant (either 0 or 1) on the preimage of each orbit of the overgroup under the fusion mapping.

This condition is fulfilled for the solutions 58 and 140. There exist 4 conjugate groups under the action of $N_G(P)$: the other three groups are admitted by the designs 51 and 139, designs 94 and 110 and designs 93 and 119 respectively.

### 4.1.2 Group $B_2$

Group $B_2$ is a perfect group of order 960 with the generators

$$
\begin{array}{l}
(1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20) \\
(1,2,7,19,6)(3,20,14,18,5)(4,8,15,13,10)(9,16,11,12,17) \\
(1,2,13,6,11)(3,17,18,5,4)(7,20,10,8,9)(12,19,14,15,16) \\
(1,2,12,3,16)(4,11,13,17,9)(5,14,18,20,10)(6,19,7,8,15)
\end{array}
$$

The fusion mapping is the following:

$$\begin{aligned}
\{1,8,9,12,16,19,20,22,50,51,53,56,64,70,71,89\} &\mapsto 1\\
\{2,67,90\} &\mapsto 2\\
\{3,54,88\} &\mapsto 3\\
\{4,6,11,31,52,66,80,83,87,91\} &\mapsto 4\\
\{5,13,15,23,24,34,37,40,41,49,59,65,75,81,92,94\} &\mapsto 5\\
\{7,10,14,17,18,29,33,42,48,60,62,69,72,73,78,86\} &\mapsto 6\\
\{21,26,27,32,43,46,57,61,68,74\} &\mapsto 7\\
\{25,55,96\} &\mapsto 8\\
\{28,58,76\} &\mapsto 9\\
\{30,63,84\} &\mapsto 10\\
\{35,38,45,77,82,93\} &\mapsto 11\\
\{36,44,85\} &\mapsto 12\\
\{39,47,79\} &\mapsto 13\\
\{95\} &\mapsto 14
\end{aligned}$$

Twelve solutions are invariant under the action of $B_2$, namely 22, 23, 25, 26, 45, 46, 78, 79, 81, 82, 95 and 96. The conjugate of $B_2$ is automorphism group of the designs 27, 28, 29, 30, 31, 32, 141, 142, 145, 146, 147 and 150.

### 4.1.3 Summary

To summarize, in total 32 designs have bigger automorphism groups. When applying the same method to the overgroups, no bigger groups are found. Therefore, $N_{N_G(A)}(P)$ acts on the designs of these resp. automorphism groups. We emphasize that the isomorphism problem has been solved without knowledge of the full automorphism groups of the designs.

In case of group $B_1$ 3 orbits of designs are obtained, each of length 4. The numbers of the representatives in Tab. 6 and 7 are 22, 23 and 45. The designs with the conjugate group as automorphism group are isomorphic to them. We visualize in Tab. 8 the 3 isomorphism types with the help of the dodecahedron. The coloured points are the elements of the representative of the block orbit.

The two designs with $B_2$ as automorphism group are isomorphic as well as all the designs with the conjugate groups as automorphism group. The representative of this isomorphism class is the design number 58 in Tab. 6 (see Fig. 5).

The other 120 designs not invariant under one of the bigger automorphism groups fall into 32 orbits. Each representative can be visualized with the help of the dodecahedron, but we only show the first one in Fig. 6.

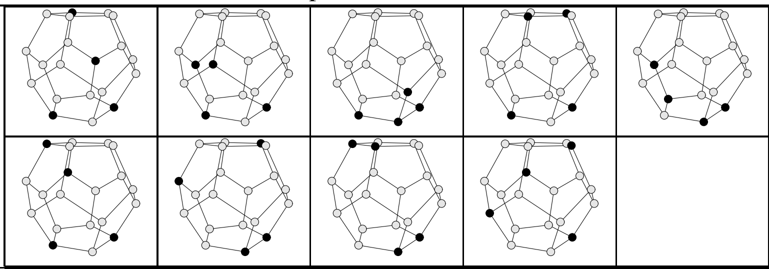
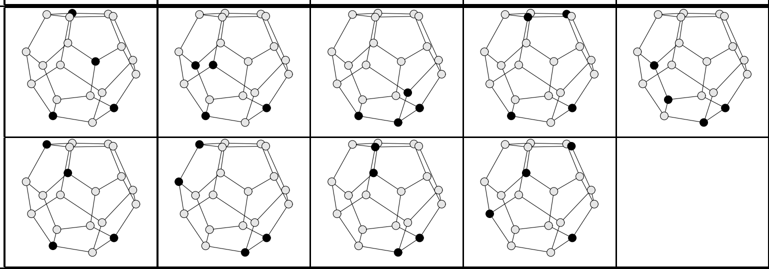
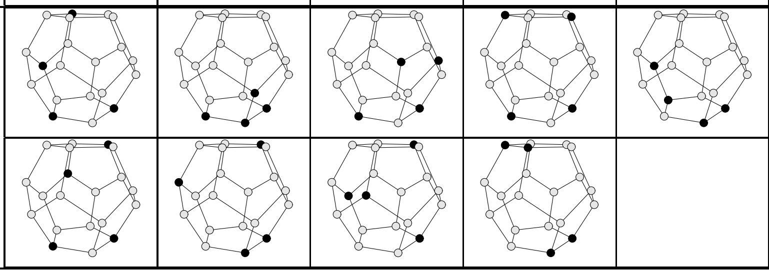
| nb. | orbit representatives on 4-sets |
|---|---|
| 22 |  |
| 23 |  |
| 45 |  |

Table 8: Visualisation of the 3 Isomorphism Types Under Group $B_1$

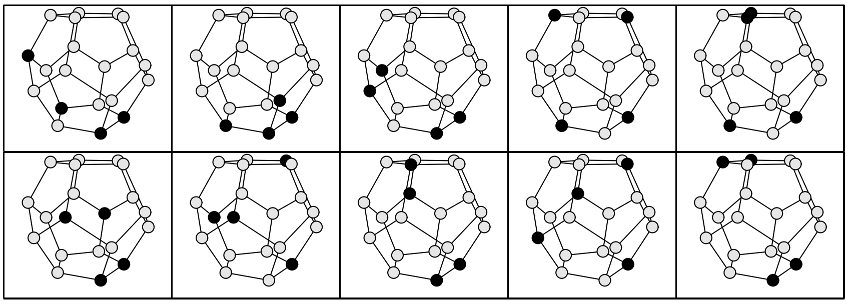


Figure 5: Visualisation of the Isomorphism Type of 3-(20, 4, 1) Under Group $B_2$

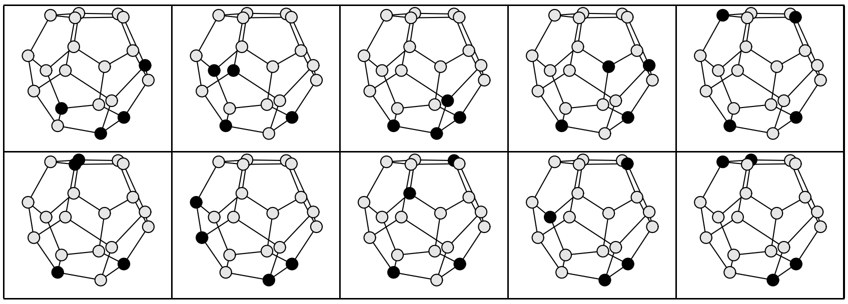


Figure 6: Visualisation of one Isomorphism Type of 3-(20, 4, 1) Under Group $A$

In total, 36 isomorphism types of designs are obtained respecting the symmetry group of the dodecahedron as prescribed automorphism group. They are listed in Tab. 9.

| Group | group order | nb. solutions | isom. types |
|---|---|---|---|
| $A$ | 60 | 120 | 32 |
| $4 \times B_1$ | 3,420 | 2 | 1 |
| $2 \times B_2$ | 960 | 12 | 3 |
| in total: | | 152 | 36 |

Table 9: Isomorphism Types of SQS(20) with Symmetry Group $A$ of the Dodecahedron

On the Webpage

```
http://www.mathe2.uni-bayreuth.de/~discreta/
SOLIDS/dodetypes.htm
```

all of them can be found visualized with the dodecahedron.

#### 4.1.4 Results for other groups of 3-(20, 4, 1) designs

We have tested some other groups as automorphism groups for SQS on 20 points. The results of the isomorphism program of DISCRETA for these groups are listed in Tab. 10.

To explain the notation: If $G$ acts on $\Omega_1$ and $H$ acts on $\Omega_2$, the group $G \times H$ acts on $\Omega_1 \times \Omega_2$ componentwise. $G+$ indicates that a fixed point has been added to the permutation representation of $G$. Finally, $G^{[l]}$ means the induced action of $G$ on $l$-sets.

| Prescribed group $A$ | group order (order of overgroups) | KM-size | # designs | # isomorphism types ($B_i$'s are overgroups) |
|---|---|---|---|---|
| $Aut(Dode) \cong \mathcal{A}_5$ | 60 (960; 3,420) | 21 × 96 | 152 | $32 \times A + 3 \times B_1 + 1 \times B_2$ |
| $Aut(\langle Dode, cent.inv.\rangle)$ | 120 | 15 × 58 | 8 | $4 \times A$ |
| $\mathcal{S}_5^{[2]} \times Id_2$ | 120 (720; 1,440) | 24 × 75 | 8 | $3 \times B_1 + 3 \times B_2$ |
| $\mathcal{S}_5^{[2]} \times C_2$ | 240 (1,440; 1,440) | 12 × 43 | 8 | $3 \times B_1 + 3 \times B_2$ |
| $\mathcal{S}_5^{[2]}$ twist $C_2$ | 120 (720; 1,440) | 15 × 57 | 16 | $3 \times B_1 + 3 \times B_2$ |
| $P\Gamma L_2(9) \times C_2$ | 2,880 | 3 × 10 | 2 | $2 \times A$ |
| $P\Gamma L_2(9) \times Id_2$ | 1,440 | 6 × 15 | 2 | $2 \times A$ |
| $P\Sigma L_2(9) \times C_2$ | 1,440 | 5 × 16 | 4 | $3 \times A$ |
| $P\Sigma L_2(9) \times Id_2$ | 720 | 10 × 26 | 4 | $3 \times A$ |
| $PGL_2(9) \times C_2$ | 1,440 | 3 × 11 | 2 | $2 \times A$ |
| $PGL_2(9) \times Id_2$ | 720 | 6 × 16 | 2 | $2 \times A$ |
| $PSL_2(9) \times C_2$ | 720 | 5 × 17 | 4 | $3 \times A$ |
| $PSL_2(9) \times Id_2$ | 360 | 10 × 27 | 4 | $3 \times A$ |
| $\mathcal{A}_5 +^{[3]}$ | 60 (360; 360) | 30 × 104 | 16 | $4 \times A + 3 \times B_1 + 3 \times B_2$ |
| $\mathcal{S}_5 +^{[3]}$ | 120 (720; 1,440) | 24 × 75 | 8 | $4 \times B_1 + 3 \times B_2$ |
| $\mathcal{A}_6^{[3]}$ | 360 | 9 × 27 | 4 | $3 \times A$ |
| $\mathcal{S}_6^{[3]}$ | 720 (1,440) | 7 × 21 | 4 | $3 \times B_1$ |
| $\left(\mathcal{S}_4^{[2]}\right)^{[3]}$ | 24 | 63 × 236 | 336 | $130 \times A$ |

Table 10: Isomorphism Classification of SQS(20) with Several Automorphism Groups

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