Graphs for Automorphisms of Famous t-Designs

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Abstract *The biplanes on up to 16 points are displayed by patterns on a graph drawing such that symmetries represent automorphisms of the designs.*

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1 Drawing t-Designs

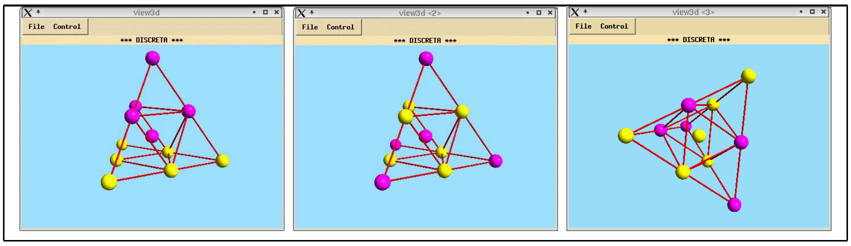
A drawing of an unlabelled graph shows that graph up to isomorphism. Colouring some vertices in a special colour then specifies an orbit of the automorphism group on subsets by a pattern. A labelling of the vertices gives for each possible position of the pattern a subset of the vertex labels. The set of vertices as a point set and some collection of such orbits as blocks in some cases form a t-design. Such a design is described by patterns on a graph drawing, see [2], [3].

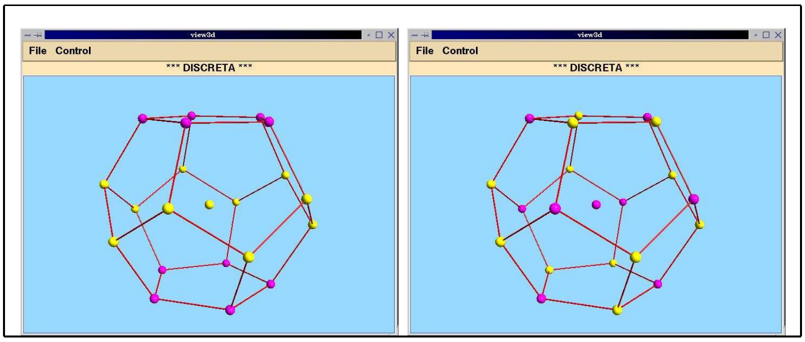
In this paper the graph consists of the points of an archimedian solid and the edges of the graph are the edges of the solid. A small example is provided by the Tetrahedron, having four points and 6 edges. Its automorphism group in 3-space induces on the set of points a permutation group isomorphic to the alternating group A_4, see [4]. All subsets of 3 points form a 2-$(4, 3, 2)$ design, the smallest biplane. This biplane admits A_4 as a group of automorphisms.

A more interesting example is obtained from the Octahedron. We construct the Octahedron from the Tetrahedron by taking as points the middle points of the edges and connecting those points by an edge that are on the border of a common face. Then we have two different kinds of faces, those that lie inside the Tetrahedron and those that are part of a face of the Tetrahedron. If we call the faces inside the Tetrahedron shaded faces and demand that symmetries should respect the shading then the automorphism group of this shaded Octahedron is again isomorphic to A_4. As an additional point we add the central point of the Octahedron. Now we take two patterns of 3-element subsets, two antipodal points and the central point as the first pattern and the 3 points on a common non-shaded face form the second pattern. We obtain a 2-$(7, 3, 1)$ design, the Fano-plane. The new drawing displays the action of A_4 of order 12 on the design. The complements of the blocks form a 2-$(7, 4, 2)$ design, the first non-trivial biplane.

The known biplanes are on the Web-pages of Gordon Royle [6], see also [1]. We give some drawings of such biplanes that display different groups of symmetries.

There is one biplane on 11 points and blocks of size 5 up to isomorphism. We can use the Tetrahedron, its embedded Octahedron that we already used before, and add the central point. The biplane is given by patterns displaying the symmetries of A_4.

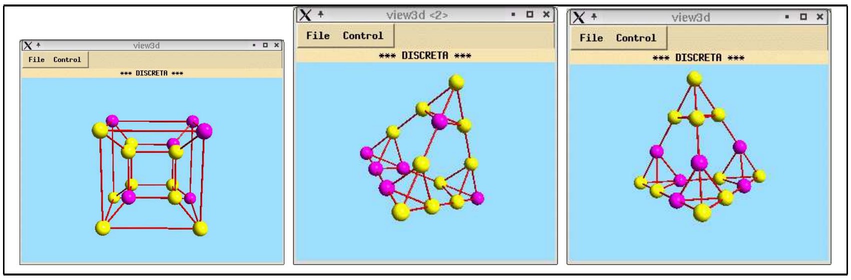




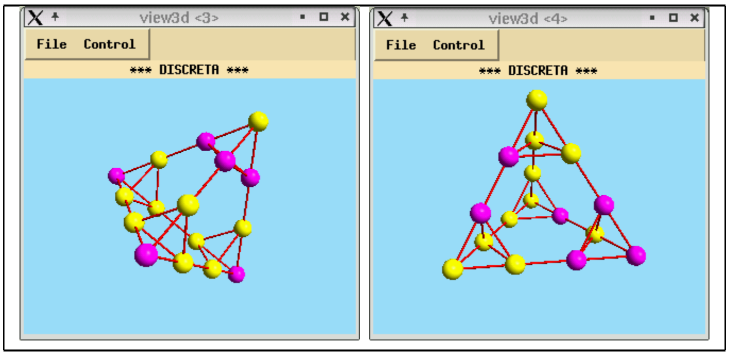
The symmetries of A_5 can be displayed by employing the Dodecahedron and again a central point. The Dodecahedron has 20 points, double as much as we need. So, we identify antipodal points by labelling the corresponding vertices with the same number. Then we still can see the symmetries of the design corresponding to the automorphism group A_5. We need 2 patterns for this design.

There are exactly 3 isomorphism types of 2-(16,6,2) designs, and a visualization of one of them is shown in [5] on a 4-dimensional Cube, see the picture. The visualizations of the other two isomorphism types of biplanes on 16 points are new.

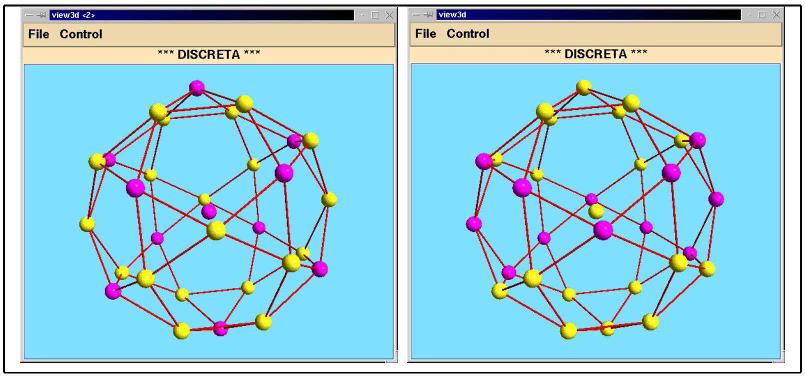
Both of them can be visualized by a Tetrahedron and its superposition by a truncated Tetrahedron. The group acting is A_4. We need two orbits of this group of lengths 12 and 4 in both cases. The isomorphism types may be identified by Hussain structures.



The second of the two designs has a large automorphism group of order 11520, being an extension $C_2^4 \cdot S_6^{[2]}$. We give two drawings of this design visualizing the symmetries that represent A_4 and A_5. The first drawing employs the Tetrahedron like the previous biplane.



We apply the same technique as in the case of the biplane on 11 points to display the symmetries corresponding to A_5. In this case, we start with the Dodecahedron and choose as new vertices the middle points of the edges.



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References

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